Review of Limit Points

Definition 2.18.

(b) $p$ is a limit point of $E$ if for every $r > 0$ there exists a point $q \neq p$ such that $q \in E \cap N_r(p)$.

(c) If $p \in E$ and $p$ is not a limit point of $E$, then $p$ is called an isolated point of $E$.

If $p$ is an isolated point of $E$, then there exists $r > 0$ such that no point $q \neq p$ satisfies $q \in E \cap N_r(p)$. That is, $E \cap N_r(p) = \{p\}$.

Limits of Functions

A major area in mathematics studies functions on subsets of $\mathbb{R}^k$ satisfying partial differential equations. It is natural and important to first study continuous functions and limits of functions.

Definition 4.1. Let $f : X \to Y$ and let $p$ be a limit point of $X$.

\[ \lim_{x \to p} f(x) = q \text{ if for every } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } d_Y(f(x), q) < \varepsilon \]

if $0 < d_X(x, p) < \delta$.

Another way to say the condition on the right is: $\forall \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that if } x \in N_\delta(p) - \{p\}, \text{ then } f(x) \in N_\varepsilon(q)$.

The left side may be thought of as: $f(x)$ approaches $q$ as $x$ approaches $p$.

Theorem 4.2. $\lim_{x \to p} f(x) = q$ if and only if for every sequence $\{p_n\}$ with $p_n \to p$ and $p_n \neq p$ for all $n$ we have $\lim_{n \to \infty} f(p_n) = q$.

Proof. $(\Rightarrow)$ Let $\{p_n\}$ be a sequence with $p_n \to p$ and $p_n \neq p$ for all $n$. We want to show that $\lim_{n \to \infty} f(p_n) = q$. So let $\varepsilon > 0$. Since $\lim_{x \to p} f(x) = q$, there exist $\delta > 0$ such that if $0 < d_X(x, p) < \delta$, then $d_Y(f(x), q) < \varepsilon$. On the other hand, since $p_n \to p$, there exists $N \in \mathbb{N}$ such that $d_X(p_n, p) < \delta$ for $n \geq N$. Moreover, since $p_n \neq p$, $d_X(p_n, p) > 0$. Combining the above, we obtain for $n \geq N$ that $d_Y(f(p_n), q) < \varepsilon$. Summarizing, we have proved that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d_Y(f(p_n), q) < \varepsilon$ for $n \geq N$. Therefore $\lim_{n \to \infty} f(p_n) = q$.

$(\Leftarrow)$ Suppose that $\lim_{x \to p} f(x) = q$ is false. We shall show that the condition on the right is false. By assumption, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists $x$ with $0 < d_X(x, p) < \delta$ and $d_Y(f(x), q) \geq \varepsilon$. In particular, by taking $\delta = \frac{1}{n}$ and calling the corresponding $x$ to be $p_n$, we have that for each $n \in \mathbb{N}$ there exists $p_n$ with $0 < d_X(p_n, p) < \delta$ and $d_Y(f(p_n), q) \geq \varepsilon$. This implies that $p_n \to p$, but $f(p_n)$ does not converge to $q$. So the condition on the right is false. \(\square\)

The right side of Theorem 4.2 says: For any sequence $p_n$ in $X - \{p\}$ limiting to $p$ we have $f(p_n)$ limits to $q$.

Corollary. If $f$ has a limit at $p$, this limit is unique.

Proof. Recall that Theorem 3.2(b) says: If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to $p$ and to $p'$, then $p = p'$. Now supposed that $\lim_{x \to p} f(x) = q$ and
\[ \lim_{x \to p} f(x) = q'. \] By assumption, \( p \) is not an isolated point of \( X \). Choose any sequence \( \{p_n\} \) with \( p_n \to p \) and \( p_n \neq p \) for all \( n \). By Theorem 4.2 we have \( \lim_{n \to \infty} f(p_n) = q \) and \( \lim_{n \to \infty} f(p_n) = q' \). Applying Theorem 3.2(b), we conclude that \( q = q' \). \( \square \)

**Definition 4.3** and **Theorem 4.4** are basic properties of functions and limits. See p. 85 of Rudin for these.

We discussed when the inputs limit to a point in the domain and the corresponding outputs limit to a point in the codomain.

**Continuous Functions**

**Definition 4.5.** \( f : X \to Y \) is continuous at \( p \) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( d_Y(f(x), f(p)) < \varepsilon \) for all \( x \) such that \( d_X(x, p) < \delta \).

In other words, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( f(x) \in N_\varepsilon(f(p)) \) for all \( x \in N_\delta(p) \).

If \( f \) is continuous at every point of \( E \), then \( f \) is called **continuous on** \( E \).

**Lemma.** If \( p \) is an isolated point of \( X \), then \( f \) is continuous at \( p \).

**Proof.** Let \( \varepsilon > 0 \). Since \( p \) is isolated, there exists \( \delta > 0 \) such that \( N_\varepsilon(p) = X \cap N_\delta(p) = \{p\} \). Hence, if \( d_X(x, p) < \delta \), then \( x = p \), so that \( d_Y(f(x), f(p)) = 0 < \varepsilon \). \( \square \)

**Theorem 4.6.** Let \( p \) be a limit point of \( X \). Then \( f \) is continuous at \( p \) if and only if \( \lim_{x \to p} f(x) = f(p) \).

**Proof (much ado about nothing).** By definition, \( \lim_{x \to p} f(x) = f(p) \) if and only if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( d_Y(f(x), f(p)) < \varepsilon \) if \( 0 < d_X(x, p) < \delta \). Since \( d_Y(f(p), f(p)) = 0 < \varepsilon \), the condition on the right is equivalent to: For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( d_Y(f(x), f(p)) < \varepsilon \) if \( d_X(x, p) < \delta \). By definition, this is equivalent to \( f \) being continuous at \( p \). \( \square \)

**Theorem 4.7.** Suppose that \( f : X \to Y \) and \( g : Y \to Z \). If \( f \) is continuous at \( p \) and if \( g \) is continuous at \( f(p) \), then \( g \circ f \) is continuous at \( p \).

**Proof.** See p. 86 of Rudin.

**Definition.** The **inverse image of a set** \( V \) by a function \( f \) is

\[ f^{-1}(V) = \{x \in X : f(x) \in V\}. \]

**Theorem 4.8.** \( f : X \to Y \) is continuous if and only if \( f^{-1}(V) \) is open in \( X \) for every open subset \( V \) of \( Y \).

**Proof.** \((\Rightarrow)\) Suppose \( f : X \to Y \) is continuous. Let \( V \subset Y \) be open. Let \( p \in f^{-1}(V) \). Then \( f(p) \in V \). Since \( V \) is open, there exists \( \varepsilon > 0 \) such that \( N_\varepsilon(f(p)) \subset V \). Since \( f \) is continuous at \( p \), there exists \( \delta > 0 \) such that \( f(x) \in N_\varepsilon(f(p)) \) for all \( x \in N_\delta(p) \). Thus \( N_\delta(p) \subset f^{-1}(N_\varepsilon(f(p))) \subset f^{-1}(V) \). We have proved that for any \( p \in f^{-1}(V) \) there exists \( \delta > 0 \) such that \( N_\delta(p) \subset f^{-1}(V) \). Therefore \( f^{-1}(V) \) is open.
Suppose \( f^{-1}(V) \) is open in \( X \) for every open subset \( V \) of \( Y \). Let \( p \in X \) and let \( \varepsilon > 0 \). We have \( f^{-1}(N_\varepsilon(f(p))) \) is open and \( p \in f^{-1}(N_\varepsilon(f(p))) \) since \( f(p) \in N_\varepsilon(f(p)) \). Therefore there exists \( \delta > 0 \) such that \( N_\delta(p) \subset f^{-1}(N_\varepsilon(f(p))) \). That is, if \( x \in N_\delta(p) \), then \( f(x) \in N_\varepsilon(f(p)) \). This proves that \( f \) is continuous at \( p \). Since \( p \) is arbitrary, \( f \) is continuous. \( \square \)

In topology, the right side of Theorem 4.8 is the definition of continuous function.

**Theorem 4.9.** Let \( f \) and \( g \) be complex continuous functions on a metric space \( C \). Then \( f + g \), \( fg \), and \( f/g \) are continuous on \( X \).

**Proof.** See p.87 of Rudin.

**Corollary.** Polynomials of several real or complex variables are continuous.

The image of a function is defined by \( f(X) = \{ f(x) : x \in X \} \).

**Theorem 4.10** is about \( \mathbb{R}^k \)-valued functions.

**Theorem 4.14.** If \( f: X \to Y \) is continuous and \( X \) is compact, then the image \( f(X) \) is compact.

**Proof.** Let \( \{ V_\alpha \}_{\alpha \in A} \) be an open cover of \( f(X) \). Let \( x \in X \). Then \( f(x) \in V_\alpha \) for some \( \alpha \in A \). Hence \( x \in f^{-1}(V_\alpha) \). Thus \( \{ f^{-1}(V_\alpha) \}_{\alpha \in A} \) is a cover of \( X \). By Theorem 4.8, since \( V_\alpha \) is open, \( f^{-1}(V_\alpha) \) is open for every \( \alpha \in A \). Since \( X \) is compact, there exists a finite subcover \( \{ f^{-1}(V_{\alpha_1}), \ldots, f^{-1}(V_{\alpha_k}) \} \) of \( X \). Since \( \{ f^{-1}(V_{\alpha_1}), \ldots, f^{-1}(V_{\alpha_k}) \} \) covers \( X \), we conclude that \( \{ V_{\alpha_1}, \ldots, V_{\alpha_k} \} \) covers \( f(X) \). \( \square \)

**Example.** One can ask what happens for \( f^{-1} \). We have the following example. Define the continuous function \( f: \mathbb{R} \to [0, 1] \) by \( f(x) = e^{-x^2} \). Note that \( f(\mathbb{R}) = (0, 1] \). So \( f^{-1}([0, 1]) = \mathbb{R} \). So if \( Y \) is compact, \( f^{-1}(Y) \) need not be compact.

By Theorem 2.41 we immediately have:

**Corollary (Theorem 4.15).** If \( f: X \to \mathbb{R}^k \) is continuous and \( X \) is compact, then the image \( f(X) \) is a closed and bounded set.

By applying Theorem 2.28 (If \( E \subset \mathbb{R} \) is nonempty, closed and bounded above (below), then \( \sup E \in E \) (\( \inf E \in E \)).) to the set \( f(X) \), we obtain:

**Corollary (Theorem 4.16).** If \( f: X \to \mathbb{R} \) is continuous and \( X \) is compact, then

1. There exists \( p \in X \) such that \( f(p) = \sup f(X) \).
2. There exists \( q \in X \) such that \( f(q) = \inf f(X) \).

**Theorem 4.17.** Suppose \( f: X \to Y \) is a continuous bijection and \( X \) is compact. Then \( f^{-1}: Y \to X \) is continuous.

**Proof.** Since \( (f^{-1})^{-1} = f \), by Theorem 4.8 it suffices to prove that \( f(V) \) is open for every open subset \( V \) of \( X \). Let \( V \subset X \) be open. Then \( V^c \) is closed. Since \( X \) is compact, by Theorem 2.35 \( V^c \) is compact. Then, by Theorem 4.14, \( f(V^c) \) is compact. By Theorem 2.34, \( f(V^c) \) is closed. Hence \( f(V^c)^c = f(V) \) is open. \( \square \)