**NOTES ON CAUCHY SEQUENCES**

**Definition 3.8.** A sequence \( \{p_n\} \) in a metric space \( X \) is called a **Cauchy sequence** if for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for all \( m, n \geq N \) we have \( d(p_m, p_n) < \varepsilon \).

**Lemma.** Cauchy sequences are bounded.

**Proof.** Let \( \{p_n\} \) be a Cauchy sequence in \( X \). Then there exists \( N \in \mathbb{N} \) such that for all \( m, n \geq N \) we have \( d(p_m, p_n) < 1 \). Then for all \( n \in \mathbb{N} \) we have

\[
d(p_n, p_N) < 1 + \max\{d(p_1, p_N), d(p_2, p_N), \ldots, d(p_{N-1}, p_N)\}.
\]

**Theorem 3.11.**

(a) In any metric space \( X \), every convergent sequence is a Cauchy sequence.

(b) If \( X \) is a compact metric space and if \( \{p_n\} \) is a Cauchy sequence in \( X \), then \( \{p_n\} \) converges to some point of \( X \).

(c) In \( \mathbb{R}^k \), every Cauchy sequence converges.

**Proof.** (a) Let \( \{p_n\} \) be a convergent sequence and let \( p \in X \) be the point to which it converges. Let \( \varepsilon > 0 \). Then there exists \( N \in \mathbb{N} \) such that if \( n \geq N \), then \( d(p_n, p) < \frac{\varepsilon}{2} \). Thus, if \( m, n \geq N \), then \( d(p_m, p) < \frac{\varepsilon}{2} \) and \( d(p_n, p) < \frac{\varepsilon}{2} \), so that

\[
d(p_m, p_n) \leq d(p_m, p) + d(p_n, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

We have proved that \( \{p_n\} \) is a Cauchy sequence.

(b) Let \( \{p_n\} \) be a Cauchy sequence in \( X \). Let \( E = \{p_1, p_2, p_3, \ldots\} \).

**Case 1.** \( E \) is finite. Let \( E = \{q_1, \ldots, q_k\} \). Let \( F_i = \{n \in \mathbb{N} : p_n = q_i\} \) for \( i = 1, \ldots, k \). Then \( \mathbb{N} = \bigcup_{i=1}^{k} F_i \). Hence there exists \( i \) such that \( F_i \) is infinite. Let \( F_i = \{n_1, n_2, \ldots\} \), where \( 1 \leq n_1 < n_2 < \cdots \). Then \( p_{n_k} = q_i \) for \( k \in \mathbb{N} \). We shall show that \( \{p_n\} \) converges to \( q_i \).

Let \( \varepsilon > 0 \). Since \( \{p_n\} \) is a Cauchy sequence, there exists \( N \in \mathbb{N} \) such that for all \( m, n \geq N \) we have \( d(p_m, p_n) < \varepsilon \). Since \( F_i \) is infinite, there exists \( n_k \in F_i \) such that \( n_k \geq N \). Then, if \( n \geq N \), we then have

\[
d(p_n, q_i) = d(p_n, p_{n_k}) < \varepsilon
\]

since \( p_{n_k} = q_i \) and \( n_k \geq N \). This completes Case 1.

**Case 2.** \( E \) is infinite. Then by Theorem 2.37 there exists a limit point \( p \) of \( E \). We shall show that \( \{p_n\} \) converges to \( p \). Let \( \varepsilon > 0 \). Since \( p \) is a limit point of \( E \), there exist \( 1 \leq n_1 < n_2 < \cdots \) such that \( d(p_{n_k}, p) < \frac{\varepsilon}{2} \) for \( k \geq 1 \). Since \( \{p_n\} \) is a Cauchy sequence, there exists \( N \in \mathbb{N} \) such that for all \( m, n \geq N \) we have \( d(p_m, p_n) < \frac{\varepsilon}{2} \). There exists \( k \geq 1 \) such that \( n_k \geq N \). Then, if \( n \geq N \), we then have (using the triangle inequality)

\[
d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

since \( n, n_k \geq N \). This completes Case 2.

(c) Let \( \{p_n\} \) be a Cauchy sequence in \( \mathbb{R}^k \). Then \( \{p_n\} \) is bounded. This implies that there exists a \( k \)-cell \( I \) such that \( p_n \in I \) for all \( n \geq 1 \). Since \( I \) is compact and since \( \{p_n\} \) is a Cauchy sequence in \( I \), by part (b) we have that \( \{p_n\} \) converges to some point of \( I \subset \mathbb{R}^k \).