Definition 3.13. A sequence \( \{ s_n \} \) of real numbers is said to be

1. increasing if \( s_{n+1} \geq s_n \) for \( n \geq 1 \);
2. decreasing if \( s_{n+1} \leq s_n \) for \( n \geq 1 \).

If \( \{ s_n \} \) is increasing or decreasing, we say that \( \{ s_n \} \) is monotonic.

Theorem 3.14. Let \( \{ s_n \} \) be a monotonic sequence. Then \( \{ s_n \} \) converges if and only if \( \{ s_n \} \) is bounded.

Proof. See p. 55 of Rudin. We shall just prove the following statement:

If \( \{ s_n \} \) is increasing and bounded, then \( \{ s_n \} \) converges to \( \sup E \), where \( E = \{ s_n : n \in \mathbb{N} \} \) is the range of \( \{ s_n \} \).

Let \( \{ s_n \} \) be an increasing and bounded sequence. Since \( E \) is bounded and nonempty, by the least upper bound property we have that \( \sup E \in \mathbb{R} \) exists. Since \( \sup E \) is an upper bound for \( E \), we have \( s_n \leq \sup E \) for all \( n \geq 1 \). Since \( \sup E \) is the least upper bound for \( E \), for any \( \varepsilon > 0 \), \( \sup E - \varepsilon \) is not an upper bound. Hence there exists \( N \geq 1 \) such that \( s_N > \sup E - \varepsilon \). Since \( \{ s_n \} \) is increasing, this implies \( s_n > \sup E - \varepsilon \) for \( n \geq N \). So we have

\[
\sup E - \varepsilon < s_n \leq \sup E \quad \text{for } n \geq N.
\]

This implies \(| s_n - \sup E | < \varepsilon \) for \( n \geq N \). We have proved that \( \{ s_n \} \) converges to \( \sup E \). \( \square \)

Definition 3.15. We say that \( \{ s_n \} \to +\infty \) (also written \( \lim_{n \to \infty} s_n = +\infty \)) if for each \( M \in \mathbb{R} \) there exists \( N \in \mathbb{N} \) such that \( s_n \geq M \) for \( n \geq N \).

Similarly, we say that \( \{ s_n \} \to -\infty \) (also written \( \lim_{n \to \infty} s_n = -\infty \)) if for each \( M \in \mathbb{R} \) there exists \( N \in \mathbb{N} \) such that \( s_n \leq M \) for \( n \geq N \).

If \( \{ s_n \} \) converges, \( \{ s_n \} \to +\infty \), or \( \{ s_n \} \to -\infty \), then we say that \( \{ s_n \} \) converges to an extended real number.

Remark. The set of extended real numbers is defined to be \( \bar{\mathbb{R}} = \mathbb{R} \cup \{ +\infty, -\infty \} \). The usual order on \( \mathbb{R} \) is extended to \( \bar{\mathbb{R}} \) by defining \( -\infty < x \) and \( x < +\infty \) for all \( x \in \mathbb{R} \) and \( -\infty < +\infty \).

Lemma. Let \( \{ s_n \} \) be a sequence of real numbers. Let \( S \subset \bar{\mathbb{R}} \) be the set of extended real number subsequential limits of \( \{ s_n \} \). Then \( S \) is nonempty.

Proof. Case 1. \( \{ s_n \} \) is bounded. Then Theorem 3.6(b) implies that there exists a convergent subsequence (converging to a real number).

Case 2. \( \{ s_n \} \) is unbounded. Then there a subsequence which converges either to \( +\infty \) or to \( -\infty \). (Exercise: Prove this.)

Again, let \( S \subset \bar{\mathbb{R}} \) denote the set of extended real number subsequential limits of a sequence \( \{ s_n \} \).

Define

\[
s^* = \sup S, \\
\ell_n = \inf S
\]

to be the upper limit and lower limit of \( \{ s_n \} \), respectively.
We use the notation:

\[ s^* = \limsup_{n \to \infty} s_n, \]
\[ s_* = \liminf_{n \to \infty} s_n. \]

Recall that \( S \subset \mathbb{R} \) is the set of extended real number subsequential limits of \( \{s_n\} \).

**Theorem 3.17.**

(a) \( s^* \in S \) and \( s_* \in S \).

(b) For each \( x > s^* \) there exists \( N \in \mathbb{N} \) such that \( s_n < x \) for \( n \geq N \). For each \( x < s_* \) there exists \( N \in \mathbb{N} \) such that \( s_n > x \) for \( n \geq N \).

Moreover, \( s^* \) and \( s_* \) are the only numbers with properties (a) and (b).

**Proof.** See p. 56 of Rudin.

**Example 3.18(a).** Let \( \{s_n\} \) be a sequence of real numbers whose range \( E \) contains all rational numbers. In other words, for each \( q \in \mathbb{Q} \) there exists \( n \geq 1 \) such that \( s_n = q \).

**Exercise:** (1) Prove that for each \( x \in \mathbb{R} \), there exists a subsequence of \( \{s_n\} \) converging to \( x \).

**Solution.** Let \( x \in \mathbb{R} \). Choose \( q_1 \in \mathbb{Q} \) such that \( q_1 \in (x, x + 1) \). Since \( \{s_n\} \) contains all rational numbers, there exists \( n_1 \in \mathbb{N} \) such that \( q_1 = s_{n_1} \). Now assume that we have chosen \( n_1, \ldots, n_{k-1} \in \mathbb{N} \). Choose \( q_k \in \mathbb{Q} \) such that \( q_k \in (x, x + \frac{1}{k}) \). Since \( \{s_n\} \) contains all rational numbers and since the interval \( (x, x + \frac{1}{k}) \) contains an infinite number of rationals, there exists \( n_k > n_{k-1} \) such that \( q_k = s_{n_k} \).

Then \( n_1 < n_2 < \cdots \) and \( |s_{n_k} - x| < \frac{1}{k} \) for each \( k \geq 1 \). Hence the subsequence \( \{s_{n_k}\} \) converges to \( x \).

(2) Prove that \( \limsup_{n \to \infty} s_n = +\infty \) and (similarly) \( \liminf_{n \to \infty} s_n = -\infty \).

**Theorem 3.20(c).**

\[ \lim_{n \to \infty} \sqrt[n]{n} = 1. \]

**Proof.** See p. 58 of Rudin.

Let \( e \) be **Euler’s number**.

**Theorem 3.31.**

\[ \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e. \]

**Proof.** See p. 64 of Rudin.

Here \( e \) is defined by the series sum \( e = \sum_{n=0}^{\infty} \frac{1}{n!} \); (infinite) **series** is a topic we consider next.