Induction

Axiom 5.1.1. Induction Principle. Let $P(n), n \in \mathbb{Z}^+$, be a sequence of statements. If

\[ P(1) \text{ and } \left( \forall k \in \mathbb{Z}^+, P(k) \Rightarrow P(k+1) \right), \]

then $\forall n \in \mathbb{Z}^+, P(n)$.

Axiom 5.4.1. Strong Induction Principle. Let $P(n), n \in \mathbb{Z}^+$, be a sequence of statements. If

\[ P(1) \text{ and } \left( \forall k \in \mathbb{Z}^+, (P(1), \ldots, P(k)) \Rightarrow P(k+1) \right), \]

then $\forall n \in \mathbb{Z}^+, P(n)$.

Proposition 1. For all integers $n \geq 4$, $n! > 2^n$.

Proof. Let $P(n)$ be the statement: $n! > 2^n$.

Base case. $P(4)$ is the statement $4! > 2^4$, which is true because $24 > 16$.

Inductive step. Suppose $k \geq 4$ satisfies $k! > 2^k$ (i.e., $P(k)$ is true). Then

\[ (k+1)! = (k+1) \cdot k! > (k+1) \cdot 2^k \geq 2 \cdot 2^k = 2^{k+1}, \]

where we used $k + 1 \geq 2$ by $k \geq 4$ for the second inequality. Hence $P(k+1)$ is true.

By mathematical induction, $n! > 2^n \forall n \geq 4$. \(\square\)

Proposition 2. For all integers $n \geq 2$, $n$ is prime or a product of primes.

Proof. Let $P(n)$ be the statement: $n$ is prime or a product of primes.

Base case. $P(2)$ is true because 2 is a prime.

Strong inductive step. Suppose $k \geq 2$ is such that each of $2, \ldots, k$ is prime or a product of primes.

Case 1. $k+1$ is prime. Then we are done.

Case 2. $k+1$ is not a prime. Then there exist integers $a$ and $b$ satisfying $k + 1 = a \cdot b$ and $1 < a, b < k + 1$, that is, $2 \leq a, b \leq k$. By the inductive hypothesis, both $a$ and $b$ are prime or a product of primes. Hence their product $a \cdot b = k + 1$ is a product of primes.

In either case, we have proved the desired conclusion that $k + 1$ is prime or a product of primes. By strong induction, we are done. \(\square\)