Functions, Injections, Surjections, and Bijections

Proposition 1. If \( f : X \to Y \) and \( g : Y \to Z \) are injections, then \( g \circ f : X \to Z \) is an injection.

Proof. Suppose that \( f : X \to Y \) and \( g : Y \to Z \) are injections. Let \( x_1, x_2 \in X \) satisfy \((g \circ f) (x_1) = (g \circ f) (x_2)\). Then \( g (f (x_1)) = g (f (x_2))\). Since \( g \) is an injection, this implies \( f (x_1) = f (x_2)\). Since \( f \) is an injection, we conclude \( x_1 = x_2\). We have proved that \( g \circ f : X \to Z \) is an injection. \(\square\)

Proposition 2. If \( f : X \to Y \) and \( g : Y \to Z \) are surjections, then \( g \circ f : X \to Z \) is a surjection.

Proof. Suppose that \( f : X \to Y \) and \( g : Y \to Z \) are surjections. Let \( z \in Z \). Since \( g \) is a surjection, there exists \( y \in Y \) such that \( g (y) = z \). Since \( f \) is a surjection, there exists \( x \in Y \) such that \( f (x) = y \). We conclude that \((g \circ f) (x) = g (f (x)) = g (y) = z\). We have proved that \( g \circ f : X \to Z \) is a surjection. \(\square\)

Definition 3. Let \( f : X \to Y \) be a function.

(i) Define \( \overrightarrow{f} : \mathcal{P} (X) \to \mathcal{P} (Y) \) by
\[
\overrightarrow{f} (A) \doteq \{ f (x) \mid x \in A \}.
\]
Given \( A \subseteq X \), i.e., \( A \in \mathcal{P} (X) \), we call \( \overrightarrow{f} (A) \) its image under \( f \).

(ii) Define \( \overleftarrow{f} : \mathcal{P} (Y) \to \mathcal{P} (X) \) by
\[
\overleftarrow{f} (B) \doteq \{ x \in X \mid f (x) \in B \}.
\]
Given \( B \subseteq Y \), i.e., \( B \in \mathcal{P} (Y) \), we call \( \overleftarrow{f} (B) \) its pre-image under \( f \).

Other notations. The image is also denoted by \( f (A) \) and the pre-image is also denoted by \( f^{-1} (A) \).

\( f : X \to Y \) is surjective if and only if \( \overrightarrow{f} (X) = Y \). \( Y \) is called the codomain and \( \overrightarrow{f} (X) \) is called the range, so \( f \) is surjective if and only if the range equals the codomain. Also, \( f \) is surjective if and only if \( \forall y \in Y \) the set \( \overrightarrow{f} (\{y\}) \) is nonempty.

\( f : X \to Y \) is injective if and only if \( \forall y \in Y \) the set \( \overleftarrow{f} (\{y\}) \) consists of at most one point.

Hence, \( f : X \to Y \) is bijective if and only if \( \forall y \in Y \) the set \( \overleftarrow{f} (\{y\}) \) consists of exactly one point.

Exercise 4. Let \( f : X \to Y \) be a function. Show that if \( C, D \in \mathcal{P} (Y) \) are such that \( C \cap D = \emptyset \), then
\[
\overleftarrow{f} (C) \cap \overleftarrow{f} (D) = \emptyset.
\]
Let \( X \) be a finite set. Define \( f : \mathcal{P} (X) \to \mathbb{Z}^\geq \), where \( \mathbb{Z}^\geq \) is the set of nonnegative integers, by \( f (A) = |A| \) is the cardinal number of \( A \). For \( k \in \mathbb{Z}^\geq \),
\[
\mathcal{P}_k (X) = \overleftarrow{f} (\{k\}) = \{ A \in \mathcal{P} (X) \mid |A| = k \}
\]
is the set of \( k \)-element subsets of \( X \). We have
\[
\mathcal{P} (X) = \bigcup_{k=0}^\infty \mathcal{P}_k (X) = \bigcup_{k=0}^{|X|} \mathcal{P}_k (X).
\]
If \( k \neq \ell \), then \( \mathcal{P}_k (X) \cap \mathcal{P}_\ell (X) = \emptyset \). This is because if \( |A| = k \) and \( |A| = \ell \), then \( k = \ell \). (This also follows from the above exercise since \( k \neq \ell \) implies \( \{k\} \cap \{\ell\} = \emptyset \).)
**Definition 5.** Given a set $X$, the **identity function** $I_X : X \to X$ is defined by $I_X(x) = x$ for all $x \in X$.

**Definition 6.** Given $f : X \to Y$, we say that a function $g : Y \to X$ is the **inverse** of $f$ if $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$. That is, $g \circ f = I_X$ and $f \circ g = I_Y$. We say that $f$ is **invertible** if it has an inverse.

**Example 7.** (a) The inverse of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is $g : \mathbb{R} \to \mathbb{R}$ defined by $g(y) = y^{1/3}$.

(b) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ does not have an inverse. On the other hand, if we restrict $f$ to $\mathbb{R}^\geq$ by defining $f|_{\mathbb{R}^\geq} : \mathbb{R}^\geq \to \mathbb{R}$ by $f|_{\mathbb{R}^\geq}(x) = f(x)$. Then the inverse of $f|_{\mathbb{R}^\geq}$ is $g : \mathbb{R} \to \mathbb{R}^\geq$ defined by $g(x) = \sqrt{x}$.

**Proposition 8.** A function $f : X \to Y$ is invertible if and only if it is a bijection.

**Proof.** (a) $f$ is invertible $\Rightarrow$ $f$ is a bijection. Suppose $f$ is invertible. Then it has an inverse $g : Y \to X$.

(i) $f$ is an injection. Suppose $x_1, x_2 \in X$ satisfy $f(x_1) = f(x_2)$. Then

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2.$$ 

Hence $f$ is an injection.

(ii) $f$ is a surjection. Let $y \in Y$. Then let $x = g(y)$. We have $f(x) = f(g(y)) = y$. Hence $f$ is a surjection.

(b) $f$ is a bijection $\Rightarrow$ $f$ is invertible. Suppose $f$ is a bijection. Define $g : Y \to X$ as follows. Let $y \in Y$. Since $f$ is a bijection there exists a unique $x \in X$ such that $f(x) = y$. $(x)$ is the unique element of $X$ such that $f(y) = \{x\}$.) Define $g(y) = x$. It is easy to see that $g$ is a well defined function.

(i) Let $x \in X$. Then $g(f(x))$ is the unique element $x'$ of $X$ such that $f(x') = f(x)$, i.e., $x' = x$ so $g(f(x)) = x$. Hence $g \circ f = I_X$.

(ii) Let $y \in Y$. Then $g(y)$ is the unique element of $x$ such that $f(g(y)) = y$. Hence $f \circ g = I_Y$. We conclude that $g$ is the inverse of $f$. □

**Exercise 9.** Define $f : \mathbb{Z}^+ \to \mathbb{Z}$ by

$$f(1) = 0,$$
$$f(2n) = n,$$
$$f(2n + 1) = -n$$

for $n \in \mathbb{Z}^+$. Find its inverse $g : \mathbb{Z} \to \mathbb{Z}^+$.

**Exercise 10.** Let $m \in \mathbb{Z}^+$. Recall that the division theorem says that for any $a \in \mathbb{Z}$ there exist unique integers $q$ and $r$ such that

$$a = mq + r \quad \text{and} \quad 0 \leq r < m.$$ 

Let $R_m = \{0, 1, \ldots, m - 1\}$. Define the function $r_m : \mathbb{Z} \to R_m$ by $r_m(a) = r$.

(i) Prove that $r_m$ is a surjection.

(ii) Prove that $r_m$ is not an injection.