Counting: Finite, Denumerable, Countable, and Uncountable Sets

Definitions 1. We say that two sets \( X \) and \( Y \) have the same cardinality if there exists a bijection \( f : X \to Y \). We write \( |X| = |Y| \).

Having the same cardinality is an equivalence relation on the ‘set of all sets’ (Russell paradox notwithstanding): (Reflexive) \( |X| = |X| \) for any set \( X \). (Symmetric) If \( |X| = |Y| \), then \( |Y| = |X| \). (Transitive) If \( |X| = |Y| \) and \( |Y| = |Z| \), then \( |X| = |Z| \).

Let \( \mathbb{N}_n = \{1, 2, \ldots, n\} \). Define the cardinality of \( \mathbb{N}_n \) to be \( n \), i.e., \( |\mathbb{N}_n| = n \). Thus \( |X| = n \) if and only if there exists a bijection \( f : \mathbb{N}_n \to X \). Define the cardinality of \( \emptyset \) to be 0, i.e., \( |\emptyset| = 0 \).

We say that \( X \) is finite if \( |X| = n \) for some \( n \in \mathbb{Z}^+ \).

We say that \( X \) is infinite if \( X \) is not finite.

We say that \( X \) is denumerable if \( X \) has the same cardinality as \( \mathbb{Z}^+ \).

It is easy to show that if \( X \) is denumerable, then \( X \) is infinite.

We have shown that \( \mathbb{Z} \) is denumerable.

We say that \( X \) is countable if \( X \) is finite or denumerable.

We say that \( X \) is uncountable if \( X \) is not countable.

Any set is exactly one of the following: finite, denumerable, or uncountable.

Any set is countable or uncountable (both not both).

Proposition 2. Let \( X \) be a finite set with \( |X| = n \), where \( n \in \mathbb{Z}^+ \). Then \( \mathcal{P}(X) \) is finite and \( |\mathcal{P}(X)| = 2^{\mathbb{P}(X)} = 2^n \).

Instead of giving a formal proof, we discuss less formally the reasons the proposition is true.

Given sets \( X \) and \( Y \), let \( \text{Fun}(X, Y) \) denote the set of all functions from \( X \) to \( Y \).

Lemma 3. For any finite set \( X \) there exists a bijection between \( \mathcal{P}(X) \) and \( \text{Fun}(X, \{0, 1\}) \).

Sketch of proof of Lemma 3. Define \( F : \mathcal{P}(X) \to \text{Fun}(X, \{0, 1\}) \) by \( F(A) = \chi_A \), where \( \chi_A \) is the characteristic function. Define \( G : \text{Fun}(X, \{0, 1\}) \to \mathcal{P}(X) \) by \( G(f) = \{ \{x \} \in X : f(x) = 1 \} \). One can show (exercise) that \( F \) and \( G \) are inverses of each other. \( \square \)

Let \( n \geq 2 \) be an integer and let \( X_1, X_2, \ldots, X_n \) be sets. Define their \( n \)-fold cartesian product by

\[
X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X_i \text{ for } i = 1, 2, \ldots, n\}.
\]

Define \( X^n \) to the \( n \)-fold cartesian product where \( X_i = X \) for each \( i = 1, 2, \ldots, n \).

Lemma 4. If \( X_1, X_2, \ldots, X_n \) are finite sets, then

\[
|X_1 \times X_2 \times \cdots \times X_n| = |X_1| \cdot |X_2| \cdots |X_n|.
\]

Sketch of proof of Lemma 4. We just prove it for \( n = 2 \). The general case can be proved by induction.

Let \( n_1 = |X_1| \) and \( n_2 = |X_2| \). Then there exist bijections \( f_1 : \mathbb{N}_{n_1} \to X_1 \) and \( f_2 : \mathbb{N}_{n_2} \to X_2 \).

Define \( f : \mathbb{N}_{n_1 n_2} \to X_1 \times X_2 \) by

\[
f(k_1 + (k_2 - 1)n_1) = (f_1(k_1), f_2(k_2)) \quad \text{for } k_1 \in \mathbb{N}_{n_1} \text{ and } k_2 \in \mathbb{N}_{n_2}.
\]

One can show that \( f \) is a bijection. \( \square \)
Lemma 5. For any finite set $X$ there exists a bijection between $\text{Fun}(X, \{0,1\})$ and $\{0,1\}^{|X|}$.

**Sketch of proof of Lemma 5.** Let $n = |X|$ and write $X = \{x_1, x_2, \ldots, x_n\}$. Define $F : \text{Fun}(X, \{0,1\}) \to \{0,1\}^n$ by

$$F(f) = (f(x_1), f(x_2), \ldots, f(x_n)).$$

Define $G : \{0,1\}^n \to \text{Fun}(X, \{0,1\})$ by $G(y_1, y_2, \ldots, y_n) = f$, where $f(x_i) = y_i$ for $i = 1, 2, \ldots, n$. One can show that $F$ and $G$ are inverses of each other. □

**Proof of Proposition 2.** As a consequence of the above,

$$2^{|X|} = \left|\{0,1\}^{|X|}\right| = |\text{Fun}(X, \{0,1\})| = \mathcal{P}(X).$$

The binomial coefficients are defined by

$$\binom{n}{r} = |\mathcal{P}_r(\mathbb{N}_n)|.$$

This is equal to the number of $r$-element subsets of a set with $n$ elements.

**Proposition 6 (Proposition 12.2.8).** For any integers $1 \leq r \leq n$ we have

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

**Proof.** Think of $\binom{n}{r}$ as the number of ways to choose a committee of $r$ members out a group of $n$ people. Call the $n$ people $1, 2, \ldots, n$. Single out the (last) person $n$. There are two (mutually exclusive) types of committees: (1) Those that contain person $n$. (2) Those that do not contain person $n$.

The number of Type (1) committees is $\binom{n-1}{r-1}$ because knowing that $n$ has to be on the committee, we have to choose $r-1$ people out the remaining $n-1$ people $1, 2, \ldots, n-1$. (A Type (1) committee consists of $r-1$ people, none which is person $n$, plus person $n$.)

The number of Type (2) committees is $\binom{n-1}{r}$ because knowing that $n$ cannot be on the committee, we have to choose $r$ people out the remaining $n-1$ people $1, 2, \ldots, n-1$. (A Type (2) committee consists of $r$ people, none which is person $n$.) □

**Proposition 7 (Theorem 12.2.10).** For any integers $0 \leq r \leq n$ we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

**Proof.** The number of ordered $r$-person committees is $n(n-1) \cdots (n-r+1)$ because there are $n$ choices for the first choice, $n-1$ choices for the second choice (one cannot choose the same person twice), ..., and $n-r+1$ choices for the $r$th choice. On the other hand, for each $r$-person committees there are $r!$ orderings. So the the number of unordered $r$-person committees is

$$\frac{n(n-1) \cdots (n-r+1)}{r!} = \frac{n!}{r!(n-r)!}.$$

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