Math 140A Midterm 2
5 Questions; 56 points total.

Instructions:
1. Write your Name, PID, and section (know your section A01 Th 5 pm or A02 Th 6 pm) on the front of your Blue Book.
2. The only things you are allowed to use are writing instruments and erasers and one page, double-sided and handwritten, of notes. (NO calculators, electronic devices, or book.)
3. Write your solutions clearly in your Blue Book and indicate the number and letter of each question.
4. Start each answer on a new page, in the same order they appear in the exam.
5. Show all of your work. No credit will be given for unsupported answers.

**Prove directly** means prove without using any theorems (from Rudin).

\( X \) denotes a **metric space** with distance function \( d \).

Given \( p \in X \) and \( r > 0 \), \( B_r(p) = \{ x \in X \mid d(x, p) < r \} \). (Denoted by \( N_r(p) \) in Rudin.)

\( \mathbb{Z}^+ = \{1, 2, 3, \ldots \} \) denotes the set of positive integers.

\( \mathbb{Q} \) denotes the set of rational numbers.

\( \mathbb{R} \) denotes the set of real numbers.

sup denotes the supremum, a.k.a., least upper bound.

inf denotes the infimum, a.k.a., greatest lower bound.

\( D \subset X \) is dense if every point of \( X \) is a limit point of \( D \) or a point of \( D \) (or both).

Copy the following table on the upper right corner of the front of your bluebook:

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1. (10 points) Give an example of a sequence \( \{x_n\} \) of integers whose set of subsequential limits is the set \( \{1, 2, 3\} \).

**Answer.** The sequence \( \{1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots\} \), where the 1’s, 2’s, and 3’s repeat infinitely many times.

**Remark.** Formally, one can define \( x_n \) as the unique remainder in \( \{1, 2, 3\} \) when we divide \( n \) by 3. That is, by the Division Theorem, there exists a unique integer \( x_n \in \{1, 2, 3\} \) such that \( n = 3q_n + x_n \) for some \( q_n \in \mathbb{Z} \). Note that \( x_n \) is the unique integer in \( \{1, 2, 3\} \) congruent to \( n \) modulo 3.

2. (12 points) Let \( (X, d) \) be a metric space.

(a) Suppose that \( \{y_n\} \) is a sequence in \( X \) such that there exists a subsequence \( \{y_{n_k}\} \) such that \( d(y_{n_k}, y_{n_l}) \geq 1 \) for all \( k, l \in \mathbb{Z} \) with \( k \neq l \). **Prove** that \( \{y_n\} \) is not a Cauchy sequence.

**Answer.** Suppose the hypothesis and suppose \( \{y_n\} \) is a Cauchy sequence. Then there exists \( N \in \mathbb{Z}^+ \) such that \( d(y_m, y_n) < 1 \) for all \( m, n \geq N \) (by taking \( \varepsilon = 1 \) in the definition of Cauchy sequence). In particular, \( d(y_{n+1}, y_N) < 1 \) since \( n+1 > N \). However this contradicts \( d(y_{N+1}, y_N) \geq 1 \).

(b) Let \( \{x_n\} \) be a Cauchy sequence in \( X \). Suppose that the subsequence \( \{x_{2n}\} \) converges to a point \( x \). **Prove** (without quoting theorems) directly that \( \{x_n\} \) converges to \( x \).

**Answer.** Let \( \varepsilon > 0 \). Since \( x_{2n} \to x \), there exists \( M \in \mathbb{Z}^+ \) such that \( d(x_{2m}, x) < \frac{\varepsilon}{2} \) for all \( m \geq M \). Since \( \{x_n\} \) is Cauchy, there exists \( N \geq M \) such that if \( n \geq 2N \), then \( d(x_n, x_{2N}) < \frac{\varepsilon}{2} \). Hence, if \( n \geq 2N \), then

\[
d(x_n, x) \leq d(x_n, x_{2N}) + d(x_{2N}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus \( \{x_n\} \) converges to \( x \).

3. (12 points) Suppose \( \{s_n\} \) is a sequence of real numbers with no peak, i.e., for each \( n \in \mathbb{Z}^+ \) there exists an integer \( m > n \) such that \( a_m > a_n \). Let \( S = \{s_n \mid n \in \mathbb{Z}^+\} \). **Prove** that \( \sup S = \limsup_{n \to \infty} s_n \).

**Hint:** You can solve this problem any way you like, but here is an outline of one approach:

(a) Suppose that \( \{s_{n_k}\} \) is a subsequence that converges to a number \( s_0 \). **Show** that \( s_0 \leq \sup S \).

**Answer.** Suppose \( s_0 > \sup S \). Then \( \varepsilon = s_0 - \sup S > 0 \). Since \( s_{n_k} \to s_0 \), there exists \( K \in \mathbb{Z}^+ \) such that \( s_{n_K} - s_0 > -\varepsilon \). Thus \( \sup S \geq s_{n_k} > s_0 - \varepsilon = \sup S \), a contradiction.

(b) **Fact:** There exists \( n_1 \in \mathbb{Z}^+ \) be such that \( s_{n_1} > \sup S - 1 \). Suppose that \( k \in \mathbb{Z}^+ \) is such that \( s_{n_j} \geq \sup S - \frac{1}{j} \) for \( 1 \leq j \leq k \). **Prove** by contradiction that there exists an integer \( n_{k+1} > n_k \) such that \( s_{n_{k+1}} \geq \sup S - \frac{1}{k+1} \).

**Answer.** Suppose \( s_n < \sup S - \frac{1}{k+1} \) for all \( n > n_k \). Let \( \bar{s} = \max\{s_1, \ldots, s_{n_k}\} \). Since \( \{s_n\} \)
has no peak, \( \bar{s} < \sup S \). Thus \( \sup S \leq \max\{\sup S - \frac{1}{k+1}, \bar{s}\} < \sup S \), a contradiction. Finally, we have proved the existence of a subsequence \( \{s_{n_k}\} \) such that \( \sup S \geq s_{n_k} \geq \sup S - \frac{1}{k} \) for all \( k \geq 1 \). This implies \( s_{n_k} \to \sup S \). By all of the above, we conclude that \( \sup S = \limsup_{n \to \infty} s_n \).

4. (12 points) Let \( \{s_n\}_{n \in \mathbb{Z}^+} \) be a sequence of real numbers converging to a number \( s \). Let

\[
\sigma_n = \frac{s_0 + s_1 + \cdots + s_{2^n}}{2^n}.
\]

Let \( \varepsilon > 0 \). Prove directly that there exists \( N \in \mathbb{Z}^+ \) such that \( |\sigma_n - s| \leq 2\varepsilon \) for all \( n \geq N \).

Hint: This is a special case of a HW problem. The proof doesn’t really change. But prove the above statement specifically and do not prove the more general result.

Answer. Since \( \lim_{n \to \infty} s_n = s \), there exists \( N \in \mathbb{Z}^+ \) such that \( |s_n - s| < \varepsilon \) for all \( n \geq N \). Let \( n \geq N \). Then

\[
|\sigma_n - s| = \left| \frac{s_0 + s_1 + \cdots + s_{2^n}}{2^n} - s \right|
\]

\[
= \left| \frac{(s_0 - s) + (s_1 - s) + \cdots + (s_{N-1} - s) + (s_N - s) + \cdots + (s_{2^n} - s)}{2^n} \right|
\]

\[
\leq \left| \frac{(s_0 - s) + (s_1 - s) + \cdots + (s_{N-1} - s)}{2^n} \right| + \left| \frac{(s_N - s) + (s_{N+1} - s) + \cdots + (s_{2^n} - s)}{2^n} \right| + \varepsilon.
\]

Clearly there exists \( N' \geq N \) such that for \( n \geq N' \),

\[
\left| \frac{(s_0 - s) + (s_1 - s) + \cdots + (s_{N-1} - s)}{2^n} \right| \leq \varepsilon.
\]

Thus, for \( n \geq N' \) we have \( |\sigma_n - s| \leq 2\varepsilon \). We conclude that \( \lim_{n \to \infty} \sigma_n = s \).

5. (10 points) Let \((X, d)\) be a metric space. Let \( D \) be a dense subset of \( X \). Let \( x \in X \) and \( r \in \mathbb{R}^+ \). Show directly that there exists \( x_0 \in D \) and \( r_0 \in \mathbb{Q}^+ \) such that \( x \in B_{r_0}(x_0) \subset B_r(x) \).

Hint: This is part of a HW problem.

Answer. Since \( D \) is dense, there exists \( x_0 \in D \) such that \( d(x_0, x) < \frac{\varepsilon}{2} \). By the Archimedean property of \( \mathbb{R} \), there exists \( r_0 \in \mathbb{Q}^+ \cap (d(x_0, x), \frac{\varepsilon}{2}) \). Since \( d(x_0, x) < r_0 \), we have \( x \in B_{r_0}(x_0) \). Let \( y \in B_{r_0}(x_0) \). Then \( d(y, x_0) < r_0 < \frac{\varepsilon}{2} \). Hence, by the triangle inequality we have \( d(y, x) \leq d(y, x_0) + d(x_0, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r \), so that \( y \in B_r(x) \). Thus \( B_{r_0}(x_0) \subset B_r(x) \).