Chapter 3 Exercise #20 on p. 82

You may start your answer as follows.

Let $\epsilon > 0$. Since $\{p_n\}$ is Cauchy, there exists $N \geq 1$ such that $d(p_n, p_m) < \frac{\epsilon}{2}$ for $n, m \geq N$.

**Remark.** In coming up with an answer, one has the choice of whether to start with "$\{p_n\}$ is Cauchy" or "some subsequence converges". I believe it is better to start with "$\{p_n\}$ is Cauchy" as above.
Chapter 3 Exercise #21 on p. 82

One approach is as follows.

For each $n$ choose $x_n \in E_n$.

Using $E_n \supset E_{n+1}$ and $\lim_{n \to \infty} \text{diam } E_n = 0$, show that $\{x_n\}$ is a Cauchy sequence.

Show that its limit point (why does this exist?), call it $x_0$, is in $E = \bigcap_{n=1}^{\infty} E_n$.

Suppose that $E$ contains two distinct points. Obtain a contradiction.
Chapter 3 Exercise #22 on p. 82

Since $G_1$ is nonempty and open, there exists $x_1 \in X$ and $r_1 > 0$ such that $E_1 \equiv B_{r_1}(x_1)$ satisfies $E_1 \subseteq G_1$.

Show that there exists $x_2 \in X$ and $r_2 \in (0, \frac{r_1}{2})$ such that $E_2 \equiv B_{r_2}(x_2)$ satisfies $E_2 \subseteq E_1$.

Explain how to continue, preferably by induction.

Define $E = \bigcap_{n=1}^{\infty} E_n$.

Explain how to use Exercise 21.
Chapter 3 Exercise #23 on p. 82
In the hint Rudin gives, which is very helpful, observe that one can switch \( n \) and \( m \). Explain what inequality this gives for the absolute value in the next display.
Chapter 3 Exercise #24 on p. 82

Let \( X \) be a metric space and let \( C \) be the set of Cauchy sequences. That is, a sequence \( p = \{p_n\} \) is in \( C \) if and only if \( p \) is Cauchy. In other words, a point \( p \) in \( C \) is the same as a Cauchy sequence \( \{p_n\} \) in \( X \).

Define the equivalence relation \( \sim \) on \( C \) by: Two Cauchy sequences \( p = \{p_n\} \) and \( q = \{q_n\} \) are equivalent, written \( p \sim q \), if \( \lim_{n \to \infty} d(p_n, q_n) = 0 \).

(a) Show that \( \sim \) is an equivalence relation on \( C \), i.e., the relation \( \sim \) is reflexive, symmetric, and transitive.

**Hint for transitivity:** Suppose \( p \sim q \) and \( q \sim r \). Use the triangle inequality. If \( \lim_{n \to \infty} d(p_n, r_n) \leq 0 \), then \( p \sim r \) (why?).

Given \( p \in C \), the equivalence class of \( p \) is \( [p] = \{ q \in C \mid q \sim p \} \), i.e., the set of all Cauchy sequences equivalent to \( p \). Let \( X^* \) be the set of equivalence classes of Cauchy sequences. That is,

\[ X^* = \{ P \mid P = [p] \text{ for some } p \in C \} \]

Given \( P, Q \in X^* \), define \( \Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n) \), where \( P = \{p_n\} \in P \) and \( Q = \{q_n\} \in Q \).

(b) (i) Show that the function \( \Delta : X^* \times X^* \to \mathbb{R}^\geq \), given by the formula above, is well defined. That is, the right-side \( \lim_{n \to \infty} d(p_n, q_n) \) is nonnegative and does not depend on the choices of \( p \in P \) and \( q \in Q \).

(ii) Suppose \( \Delta(P, Q) = 0 \). Show that \( P = Q \).

(iii) Show that the triangle inequality for \( \Delta \) follows from the triangle inequality for \( d \).

**Remark on (b)(i).** Suppose \( p, p' \in P \) and \( q, q' \in Q \), where \( p' = \{p'_n\} \) and \( q' = \{q'_n\} \). We need to show that \( \lim_{n \to \infty} d(p_n, q_n) = \lim_{n \to \infty} d(p'_n, q'_n) \). By hypothesis, \( \lim_{n \to \infty} d(p_n, p'_n) = 0 \) and \( \lim_{n \to \infty} d(q_n, q'_n) = 0 \). Use the triangle inequality.

(c) Show that \( X^* \) is complete. Let \( \{ P_i \} \) be a Cauchy sequence in \( X^* \). (Note that \( \{ P_i \} = \{ P_1, P_2, P_3, \ldots \} \), where each \( P_i \) is itself a Cauchy sequence. So \( \{ P_i \} \) is a Cauchy sequence of Cauchy sequences.)

Show that \( \{ P_i \} \) converges to some \( P \in X^* \). That is, there exists \( P \in X^* \) such that \( \Delta(P, P_i) \to 0 \) as \( i \to \infty \).

**Hint:** For each \( i \in \mathbb{Z}^+ \), we may write \( P_i = [p_i] \), where \( p_i = \{p_i(1), p_i(2), p_i(3), \ldots \} \) is a Cauchy sequence. Define \( P = [p] \), where \( p = \{p(1), p(2), p(3), \ldots \} \). That is, \( P \) is the equivalence class of the Cauchy sequence \( p \) whose \( n \)-th term is the same as the \( n \)-th term of \( p_n \).

(i) Show that \( p \) is a Cauchy sequence, so that \( P \in X^* \).

(ii) Show that \( \{ P_i \} \) converges to \( P \). That is, \( \Delta(P, P_i) \to 0 \). Note that, for each \( i \), \( \Delta(P_i, P) = \lim_{n \to \infty} d(p_i(n), p_n(n)) \).

Define \( \varphi : X \to X^* \) by \( \varphi(p) = P_p \), where \( P_p \upharpoonright \{p\} \) for \( p = \{p, p, \ldots \} \) the constant sequence.
(d) Show that for any $p, q \in X$, we have $\Delta(P_p, P_q) = d(p, q)$. Why does this imply that $\varphi$ is an injection (a.k.a., one-to-one)?

(e) Show that $\varphi(X)$ is dense in $X^*$.

Hint: Let $P = \{p\} \in X^*$, where $p = \{p_n\}$. Define $p_n = \{p_n, p_n, p_n, \ldots\}$ the constant sequence. Let $P_n = \{p_n\} \in X^*$ Show that $\lim_{n \to \infty} \Delta(P_n, P) = 0$.

followup discussions for lingering questions and comments

Anonymous 16 hours ago
For (c), what does it mean by saying a sequence of equivalent classes converges to a specific equivalent class?

Bennett Chow 13 hours ago
I've edited the post to clarify this. See the third line of (c).

I corrected the hint for (e).

I've added a few other edits.