Math 109 HW 4. Due date: Friday October 28 at 3 pm

Exercises: # 11.3, 11.4, 12.5, 12.6
Problems III: # 5, 12, 13, 15.

Hints

Problems III #5. Prove that a finite non-empty set of real numbers has a minimum element. [This is the part of the proof of Proposition 11.2.3 left as an exercise.]

Hint. Mimic the proof by induction on p. 140 of Eccles that a finite non-empty set of real numbers has a maximum element.

Formally, \( P(n) \), where \( n \in \mathbb{Z}^+ \), is the statement that any finite set with cardinality \( n \) has a minimum element.

Problems III #12. Suppose that there is an injection \( f : \mathbb{Z}^+ \rightarrow X \). Prove by contradiction that \( X \) is an infinite set. [Use Corollary 11.1.1 noting that, for any \( n \geq 1 \), \( f \) restricts to give an injection \( \mathbb{N}_{n+1} \rightarrow X \).]

Hint. Corollary 11.1.1 says: Suppose \( X \) and \( Y \) are non-empty finite sets. If there exists an injection \( f : X \rightarrow Y \), then \( |X| \leq |Y| \).

Suppose \( X \) is a finite set. Then there exists \( n_0 \in \mathbb{Z}^+ \) and an injection \( g : X \rightarrow \mathbb{N}_{n_0} \). Use the book hint and compose two functions.

Problems III #13. Find all the divisors of 126 and 180 and hence find the greatest common divisor (126, 180).

Hint. 126 = 2 \cdot 3^2 \cdot 7 and 180 = 2^2 \cdot 3^2 \cdot 5.

Problems III #15. Prove the induction principle from the well-ordering principle (see Example 11.2.2(c)). [Prove the induction principle in the form of Axiom 7.5.1 by contradiction.]

Hint. Axiom 7.5.1 says: Suppose that \( A \) is a subset of \( \mathbb{Z}^+ \). Then \( A = \mathbb{Z}^+ \) if

(i) \( 1 \in A \), and
(ii) \( \forall k \in \mathbb{Z}^+ (k \in A \Rightarrow k + 1 \in A) \).

The well-ordering principle says: Any non-empty set of positive integers has a minimum element.

Let \( S = A^c = \mathbb{Z}^+ - A \). Suppose that \( S \) is non-empty. Etc.

Exercise #11.3. Easy.

Exercise #11.4. Prove that, if \( a \) and \( b \) are non-zero integers with \( \gcd(a, b) = d \), then the integers \( a/d \) and \( b/d \) are coprime.

Hint. See the solution at the back of Eccles.

Exercise #12.5. By equating coefficients of \( x^n \) in \( (1+x)^2n = (1+x)^n(1+x)^n \) prove that \( \binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2 \).

Hint. We have \( (1+x)^n = \sum_{i=0}^{n} a_i x^i \), where \( a_i = \binom{n}{i} \). So
\[(1 + x)^n(1 + x)^n = \sum_{i=0}^{n} a_i x^i \cdot \sum_{j=0}^{n} a_j x^j = \sum_{k=0}^{2n} \sum_{i+j=k} a_i a_j x^k, \text{ where } i, j \geq 0 \text{ in the last sum.}\]

We may rewrite this as \[\sum_{k=0}^{2n} \left( \sum_{i=0}^{k} a_i a_{k-i} \right) x^k.\]

On the other hand, \[(1 + x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k.\]

Equate the \(k = n\) coefficient in the expansions for \((1 + x)^{2n}\) and \((1 + x)^n(1 + x)^n\).

**Exercise #12.6.** Prove that the product of any \(n\) consecutive positive integers is divisible by \(n!\)

**Hint.** Let \(p\) be the product of any \(n\) consecutive positive integers. Then there exists \(k \in \mathbb{Z}^+\) such that \(n = (k + 1) \cdots (k + n)\). Then \(n = \frac{(k+n)!}{k!}\). Etc.