HW 4. Due date: Friday October 28 at 2 pm

1. Chapter 2 Exercise #22 on p. 45. **Hint:** You may use that \( \mathbb{Q} \) is dense in \( \mathbb{R} \).

**Answer.** A subset \( E \) of a metric space \( X \) is **dense** if \( X = E \cap E' \). Equivalently, \( E \) is dense in \( X \) if for any \( q \in X \) and \( r > 0 \) there exists \( p \in E \) such that \( d(p, q) < r \). A metric space is called **separable** if it contains a countable dense subset.

Given \( x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in \mathbb{R}^k \),

\[
d(x, y) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}.
\]

Let \( y \in \mathbb{R}^k \) and \( r > 0 \). Since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), for each \( i = 1, 2, \ldots, k \), there exists \( x_i \in \mathbb{Q} \) such that \( |x_i - y_i| < \frac{r}{\sqrt{k}} \). Then \( x \divides (x_1, \ldots, x_k) \in \mathbb{Q}^k \) satisfies

\[
d(x, y)^2 = \sum_{i=1}^{k} (x_i - y_i)^2 < \sum_{i=1}^{k} \frac{r^2}{k} = r^2,
\]

i.e., \( d(x, y) < r \). Hence \( \mathbb{Q}^k \) is a dense subset of \( \mathbb{R}^k \). Finally, since \( \mathbb{Q} \) is countable and since the cartesian product of countable sets is countable, \( \mathbb{Q}^k \) is countable. Therefore \( \mathbb{R}^k \) is separable. \( \Box \)

2. (a) Describe a sequence \( \{a_n\}_{n=1}^{\infty} \) of positive integers whose set of subsequential limits is the set \( \mathbb{Z}^+ \).

**Answer.** Let \( \{a_n\}_{n=1}^{\infty} \) be the sequence

\[
1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots, 1, 2, \ldots, k, 1, \ldots
\]

Then each positive integer \( k \) occurs an infinite number of times in the sequence. Hence each \( k \in \mathbb{Z}^+ \) is a subsequential limit of \( \{a_n\}_{n=1}^{\infty} \).

More formally, one can define \( f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) by

\[
f(n) = a_n = n - \frac{j(j-1)}{2} \quad \text{if} \quad \frac{j(j-1)}{2} < n \leq \frac{(j+1)j}{2} \quad \text{for} \quad j \in \mathbb{Z}^+.
\]

Note that \( \frac{(j+1)j}{2} - \frac{j(j-1)}{2} = j \).

So \( f^{-1}(k) = \left\{ \frac{j(j-1)}{2} + k \mid j \geq k \right\} \), which is an infinite set for each \( k \in \mathbb{Z}^+ \).

Correspondingly, for each \( k \in \mathbb{Z}^+ \),

\[
\left\{ a_{\frac{j(j-1)}{2} + k} \right\}_{j=k}^{\infty}
\]

is a subsequence of \( \{a_n\}_{n=1}^{\infty} \) with each term equal to \( k \), so its limit is \( k \).

**Discussion.** The following explains how the formula for \( a_n \) was obtained:

\[
\begin{array}{cccccc}
1 & 1, 2 & 1, 2, 3 & \ldots, 1, \ldots, j-1 & 1, 2, \ldots, j, 1, \ldots \\
1 & 2 & 3 & \ldots, j-1 & j & \\
1+2+3+\cdots+(j-1) = \frac{j(j-1)}{2} & (j+1)j & \\
\end{array}
\]
(b) Let \( f : \mathbb{Z}^+ \to \mathbb{Q} \) be a bijection. Define the sequence \( \{b_n\}_{n=1}^\infty \) by \( b_n = f(a_n) \). What is the set of subsequential limits of \( \{b_n\}_{n=1}^\infty \)? \textbf{Hint: Consider Theorem 3.7.}

\textbf{Answer.} Since \( f \) is surjective and since each positive integer occurs an infinite number of times in \( \{a_n\}_{n=1}^\infty \), each rational number occurs an infinite number of times in \( \{b_n\}_{n=1}^\infty \), so the set \( E^* \) of subsequential limits of \( \{b_n\}_{n=1}^\infty \) contains \( \mathbb{Q} \). By Theorem 3.7, \( E^* \) is a closed set. Hence \( E^* \) contains \( \mathbb{Q} \), which is equal to \( \mathbb{R} \) since \( \mathbb{Q} \) is a dense subset of \( \mathbb{R} \). We conclude that \( E^* = \mathbb{R} \).

Alternatively (more formally), let \( q \in \mathbb{Q} \). Since \( f \) is surjective, there exists \( m \in \mathbb{Z}^+ \) such that \( f(m) = q \). Then \( \left\{ a_{(j-1)+m} \right\}_{j=m}^\infty \) is a subsequence of \( \{a_n\} \) that is the constant \( m \). So the corresponding subsequence \( \left\{ b_{(j-1)+m} \right\}_{j=m}^\infty \) of \( \{b_n\} \) is the constant \( q \). Hence each \( q \in \mathbb{Q} \) is a subsequential limit of \( \{b_n\} \). The rest of the answer is the same as the above.

(c) Show that there exists a sequence \( \{c_n\}_{n=1}^\infty \) of real numbers whose set of subsequential limits is the interval \([0, 1] \).

\textbf{Answer.} The set \( \mathbb{Q} \cap [0, 1] \) is countable. Hence there exists a bijection \( g : \mathbb{Q} \to \mathbb{Q} \cap [0, 1] \). Since \( b_n \in \mathbb{Q} \) for all \( n \in \mathbb{Z}^+ \), the sequence \( \{c_n\}_{n=1}^\infty \) given by \( c_n = g(b_n) \) is well defined. Since \( g \) is a surjection, each element of \( \mathbb{Q} \cap [0, 1] \) occurs an infinite number of times in \( \{c_n\}_{n=1}^\infty \), so the set \( E^* \) of subsequential limits of \( \{c_n\}_{n=1}^\infty \) contains \( \mathbb{Q} \cap [0, 1] \). Since \( \mathbb{Q} \cap [0, 1] \) is dense in \([0, 1] \) and \( E^* \) is closed, \( E^* \) contains \([0, 1] \). On the other hand, since \([0, 1] \) is closed and since \( c_n \in [0, 1] \) for all \( n \in \mathbb{Z}^+ \), \( E^* \subset [0, 1] \). We conclude that \( E^* = [0, 1] \).

3. Let \( (X,d) \) be a metric space with the property that for every \( p \in X \) and \( r > 0 \), \( B_r(p) = \{ x \in X | d(x,p) \leq r \} \) is compact. Let \( \{p_n\}_{n=1}^\infty \) be a sequence of points in \( X \).

(a) Prove that if \( \{p_n\}_{n=1}^\infty \) has a bounded subsequence, then \( \{p_n\}_{n=1}^\infty \) has a convergent subsequence.

\textbf{Answer.} By hypothesis, there exists a subsequence \( \{p_{n_k}\}_{k=1}^\infty \) that is bounded. That is, given \( p \in X \), there exists \( r > 0 \) such that \( p_{n_k} \in B_r(p) \) for all \( k \geq 1 \). Since \( \{p_{n_k}\}_{k=1}^\infty \) is a sequence in a compact set \( B_r(p) \), it has a subsequence \( \{p_{n_{k\ell}}\}_{\ell=1}^\infty \) which converges. Finally, \( \{p_{n_{k\ell}}\}_{\ell=1}^\infty \) is a subsequence of \( \{p_n\}_{n=1}^\infty \) since any subsequence of a sequence is a subsequence (because the composition of two strictly increasing functions is strictly increasing: if \( b_n = a_{g(n)} \) and \( c_n = b_{h(n)} \), where \( g, h : \mathbb{Z}^+ \to \mathbb{Z}^+ \) are strictly increasing, then \( c_n = a_{g(h(n))} \) and \( g \circ h : \mathbb{Z}^+ \to \mathbb{Z}^+ \) is strictly increasing).

(b) Choose \( q \in X \). Let \( N_r = \{ n \in \mathbb{Z}^+ | d(p_n,q) \leq r \} \). Prove that if there exists \( r > 0 \) such that \( N_r \) is an infinite set, then \( \{p_n\}_{n=1}^\infty \) has a convergent subsequence.

\textbf{Answer.} By hypothesis, there exists \( r > 0 \) such that \( N_r \) is an infinite set. Then \( \{p_n\}_{n \in N_r} \) is a subsequence of \( \{p_n\}_{n=1}^\infty \) contained in \( B_r(p) \). Since \( \{p_n\}_{n \in N_r} \) is a bounded subsequence, by part (a) we conclude that \( \{p_n\}_{n=1}^\infty \) has a convergent subsequence.
Remark. Since $N_r \subset \mathbb{Z}^+$ is infinite, we may write $N_r = \{n_1, n_2, n_3, \ldots \}$, where $1 \leq n_1 < n_2 < n_3 < \cdots$. Then $\{p_n\}_{n \in N_r}$ is the same as $\{p_{nk}\}_{k \in \mathbb{Z}^+}$.

4. Chapter 2 Exercise #23 on p. 45.

Answer. A collection $\{V_\alpha\}_{\alpha \in A}$ of open subsets of $X$ is a base for $X$ if for each $x \in X$ and open set $G$ with $x \in G$, there exists $\alpha \in A$ such that $x \in V_\alpha \subset G$.

Let $D$ be a countable dense subset of $X$. Define the collection of open subsets:

$$B = \{B_r(x) \mid x \in D \text{ and } r \in \mathbb{Q}^+\}.$$  

(1) In other words, $B = \{B_r(x) \}_{(x,r) \in D \times \mathbb{Q}^+}$. Since $D$ and $\mathbb{Q}^+$ are both countable, $D \times \mathbb{Q}^+$ is countable. Hence $B$ is a countable collection.

(2) Let $G$ be an open set and let $x \in G$. Then there exists $r \in \mathbb{R}^+$ such that $B_r(x) \subset G$.

Claim. There exist $r' \in \mathbb{Q}^+$ and $x' \in D$ such that $x \in B_{r'}(x') \subset B_r(x)$.

Proof of the claim. Since $D$ is dense, there exists $x' \in D$ such that $d(x', x) = s < \frac{r}{2}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ (Theorem 1.20(b)), there exists $r' \in \mathbb{Q}$ such that $s < r' < \frac{r}{2}$. Since $d(x', x) = s < r'$, $x \in B_{r'}(x')$.

Now let $y \in B_{r'}(x')$. Then

$$d(y, x) \leq d(y, x') + d(x', x) < r' + r' < r.$$  

Hence $y \in B_r(x)$. This proves the claim.

Finally, by the claim, $x \in B_{r'}(x') \subset B_r(x) \subset G$, where $x' \in D$ and $r' \in \mathbb{Q}^+$. This proves that $B$ is a base for $X$.

5. Chapter 2 Exercise #24 on p. 45.

Answer. Let $\delta > 0$. Pick $x_1 \in X$. Given $j \in \mathbb{Z}^+$ and having chosen $x_1, \ldots, x_j \in X$. Choose, if possible, $x_{j+1} \in X$ so that $d(x_i, x_{j+1}) \geq \delta$ for all $i = 1, \ldots, j$.

Claim. This process must stop after a finite number of steps. That is, there exists $J \in \mathbb{Z}^+$ such that for every $x \in X$ we have $d(x, x) < \delta$ for some $i = 1, \ldots, J$.

Proof of the claim. Suppose the claim is false. Then there exists a sequence $\{x_j\}_{j \in \mathbb{Z}^+}$ such that $d(x_i, x_j) \geq \delta$ for all $1 \leq i < j < \infty$. This condition is equivalent to $d(x_i, x_j) \geq \delta$ for all $i \neq j$. Since the sequence $\{x_j\}_{j \in \mathbb{Z}^+}$ is pairwise disjoint, i.e., $x_i \neq x_j$ for $i \neq j$, the set $S = \bigcup_{j \in \mathbb{Z}^+} \{x_j\}$ is infinite.

Suppose $S$ has a limit point, call it $x_0$. Then there exists $i$ and $j$ with $i \neq j$ such that $d(x_i, x_0) < \frac{\delta}{2}$ and $d(x_j, x_0) < \frac{\delta}{2}$ (there are in fact an infinite number of such points). But this implies

$$d(x_i, x_j) \leq d(x_i, x_0) + d(x_j, x_0) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

a contradiction. Hence the infinite set $S = \bigcup_{j \in \mathbb{Z}^+} \{x_j\}$ does not have a limit point, contradicting the hypothesis. This proves the claim.
By the claim, for each $\delta > 0$ there exists $J \in \mathbb{Z}^+$ such that $X = \bigcup_{j=1}^{J} B_{\delta}(x_j)$.

By taking $\delta = 1/n$, for each $n \in \mathbb{Z}^+$ there exist $J_n \in \mathbb{Z}^+$ and points $x_{n,1}, \ldots, x_{n,J_n}$ such that

$$X = \bigcup_{j=1}^{J_n} B_{1/n}(x_{n,j}).$$

**Claim.** $\mathcal{B} = \{B_{1/n}(x_{n,j})\}_{1 \leq j \leq J_n, n \in \mathbb{Z}^+}$ is a countable base for $X$.

**Proof of the claim.** $\mathcal{B}$ is a countable collection since it is the index set $\{\{n,j\} | 1 \leq j \leq J_n, n \in \mathbb{Z}^+\}$ is the countable union of finite sets.

Let $x \in X$ and let $G$ be an open subset of $X$ such that $x \in G$. For each $n \in \mathbb{Z}^+$ there exists $j \in \mathbb{Z}^+$ with $1 \leq j \leq J_n$ such that $x \in B_{1/n}(x_{n,j})$. Then $B_{1/n}(x_{n,j}) \subset B_{2/n}(x)$; proof: if $y \in B_{1/n}(x_{n,j})$, then $d(y, x) \leq d(y, x_{n,j}) + d(x, x_{n,j}) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$. Since $G$ is open and $x \in G$, there exists $r > 0$ such that $B_r(x) \subset G$. By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{Z}^+$ such that $\frac{2}{n} < r$. We conclude that $x \in B_{1/n}(x_{n,j})$ and $B_{1/n}(x_{n,j}) \subset B_{2/n}(x) \subset B_r(x) \subset G$. Thus $\mathcal{B}$ is a base for $X$. This proves the claim.

6. **Chapter 2 Exercise #25 on p. 45.**

**Answer.** Let $(K, d)$ be a compact metric space. Let $n \in \mathbb{Z}^+$. Then $\{B_{1/n}(x)\}_{x \in K}$ is an open cover of $K$. Since $K$ is compact, there exists a finite subset $A_n$ of $K$ such that $\{B_{1/n}(x)\}_{x \in A_n}$ covers $K$. Let $\mathcal{B} = \{B_{1/n}(x)\}_{x \in A_n, n \in \mathbb{Z}^+}$. By Exercise #24, $\mathcal{B}$ is a countable base for $K$. 

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