1. Chapter 3, p. 82 #20. Suppose \( \{p_n\} \) is a Cauchy sequence in a metric space \( X \), and some subsequence \( \{p_{n_i}\} \) converges to a point \( p \in X \). Prove that the full sequence \( \{p_n\} \) converges to \( p \).

**Solution.**

Take any \( \epsilon > 0 \). Since \( \{p_n\} \) is Cauchy, there exists \( N \in \mathbb{Z}^+ \) such that if \( n, m \geq N \) then \( d(p_n, p_m) < \frac{\epsilon}{2} \).

Also, because \( p_{n_j} \) converges to \( p \), there exists \( J \in \mathbb{Z}^+ \) such that if \( j \geq J \) then \( d(p_{n_j}, p) < \frac{\epsilon}{2} \). Now take any \( j \geq \max(J, N) \). Then \( n_j \geq j \geq J \) (since \( n_1 < n_2 < \ldots \)), so that \( d(p_{n_j}, p) < \frac{\epsilon}{2} \). If \( n \geq N \), then

\[
d(p_n, p) \leq d(p_n, p_{n_j}) + d(p_{n_j}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

where the inequality \( d(p_n, p_{n_j}) < \frac{\epsilon}{2} \) is used and follows from \( n_j \geq j \geq N \).

2. Chapter 3, p. 82 #21. Prove the following analogue of Theorem 3.10(b): If \( \{E_n\} \) is a sequence of closed nonempty and bounded sets in a complete metric space \( X \), if \( E_n \supseteq E_{n+1} \), and if \( \lim_{n \to \infty} \text{diam } E_n = 0 \), then \( \bigcap_{n=1}^{\infty} E_n \) consists of exactly one point.

**Solution.**

Since \( E_n \) is nonempty for any positive integer \( n \), take any \( x_n \in E_n \). This creates a sequence \( x_1, x_2, \ldots \) in \( X \). First we show that this sequence is Cauchy. Take any \( \epsilon > 0 \). Then since \( \lim_{n \to \infty} \text{diam } E_n = 0 \), there exists \( N \in \mathbb{Z}^+ \) such that if \( n \geq N \) then \( \text{diam } E_n < \epsilon \). If \( n, m \geq N \), then \( x_n, x_m \) are both in \( E_N \), so \( d(x_n, x_m) \leq \text{diam } E_N < \epsilon \).

Since \( X \) is complete, there exists \( x \in X \) such that \( \lim_{n \to \infty} x_n = x \). If for some positive integer \( m \), \( x \notin E_m \), then because \( E_m \) is closed there exists \( r > 0 \) such that \( B_r(x) \cap E_m = \emptyset \). Since \( x_n \) converges to \( x \), there exists \( N \in \mathbb{Z}^+ \) such that if \( m \geq N \) then \( x_n \in B_r(x) \). If we take \( n = \max(N, m) \), then \( x_n \in B_r(x) \cap E_n \), and \( E_n \subseteq E_m \), so \( x_n \in B_r(x) \cap E_m \), contradiction. Therefore \( x \in E_m \) for all positive integers \( m \), so \( x \in \bigcap_{n=1}^{\infty} E_n \). That is, \( \bigcap_{n=1}^{\infty} E_n \) is nonempty.

Now suppose that \( y \in \bigcap_{n=1}^{\infty} E_n \). Then for all positive integers \( n \), \( 0 \leq d(y, x) \leq \text{diam } E_n \). Since \( \lim_{n \to \infty} \text{diam } E_n = 0 \), we have \( d(y, x) = 0 \) hence \( y = x \).

Remark. To justify the last sentence from the above paragraph, we can proceed as follows: Take any \( \epsilon > 0 \). Then there exists \( N \in \mathbb{Z}^+ \) such that \( \text{diam } E_n < \epsilon \) if \( n \geq N \). Since \( x, y \in E_N \), \( 0 \leq d(y, x) \leq \text{diam } E_N < \epsilon \). So we’re in the situation where for all \( \epsilon > 0 \), \( 0 \leq d(y, x) < \epsilon \). If \( d(y, x) \neq 0 \), then taking \( \epsilon = d(y, x) \) contradicts this condition on \( d(y, x) \).

3. Chapter 3, p. 82 #22. Suppose \( X \) is a nonempty complete metric space, and \( \{G_n\} \) is a sequence of dense open subsets of \( X \). Prove Baire’s theorem, namely, that \( \bigcap_{n=1}^{\infty} G_n \) is not empty.

**Solution.**
Since $X$ is nonempty, each $G_n$ is nonempty (otherwise $G_n$ would also be an empty set, contradicting density of $G_n$). First take $x_1 \in G_1$. Since $G_1$ is open, there exists $r > 0$ such that $B_r(x_1) \subset G_1$. Taking $r_1 = \min(\frac{1}{2}, r)$, we have $B_{r_1}(x_1) \subset G_1$. Now we note that the closure of $B_{r_1/2}(x_1)$ is contained in $B_r(x_1)$. To see this, if $y \notin B_{r_1}(x_1)$, then $d(y, x_1) \geq r_1$, so that $B_{r_1/2}(y) \cap B_{r_1/2}(x_1)$ is empty, therefore $y \notin B_{r_1/2}(x_1)$. Note that $B_{r_1/2}(y) \cap B_{r_1/2}(x_1)$ is empty because if $z \in B_{r_1/2}(y) \cap B_{r_1/2}(x_1)$ then by the triangle inequality, $d(x, y) \leq d(x, z) + d(z, y) < \frac{r_1}{2} + \frac{r_1}{2} = r_1$, contradicting $d(x, y) \geq r_1$. Note also that $r_1 < 1$.

Take $E_1$ to be the closure of $B_{r_1/2}(x_1)$. Now since $G_2$ is dense in $X$, $B_{r_1/2}(x_1) \cap G_2$ is also nonempty, so take a point $x_2 \in B_{r_1/2}(x_1) \cap G_2$. Since a finite intersection of open sets is again open, we can find $r > 0$ such that $B_r(x_2) \subset B_{r_1/2}(x_1) \cap G_2$. Take $r_2 = \min(r, \frac{1}{2r})$. As before, the closure of $B_{r_2}(x_1)$ is contained in $B_r(x_2)$. We let $E_2$ to be this closure of $B_{r_2}(x_2)$.

Inductively then build $E_n$ as the closure of $B_{r_n}(x_n)$ where $r_n = \min(r, \frac{1}{2r})$ and where $B_r(x_n) \subset B_{r_{n-1}/2}(x_{n-1}) \cap G_n$ and where $x_n \in B_{r_{n-1}/2}(x_{n-1}) \cap G_n$ can be taken by denseness of $G_n$.

We have thus a nested sequenced of nonempty closed sets $E_1 \supset E_2 \supset E_3 \supset \cdots$ such that $E_n \subset G_n$ for all $n$. Also, $\operatorname{diam}(E_n) \leq \frac{1}{n}$ and $X$ is complete, so we’re in the situation of problem #21, pg. 82 of Rudin. Therefore $\bigcap_{n=1}^{\infty} E_n$ is nonempty, and $\bigcap_{n=1}^{\infty} E_n \subset \bigcap_{n=1}^{\infty} G_n$, so the latter is also nonempty.

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4. Chapter 3, p. 82 #23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space $X$. Show that the sequence $\{d(p_n, q_n)\}$ converges.

**Solution.**

For any $m, n$, we have $d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$. Likewise, $d(p_m, q_m) \leq d(p_m, p_n) + d(p_n, q_n) \leq d(p_m, p_n) + d(p_n, q_m) + d(q_m, q_n)$. Therefore $d(p_n, q_n) - d(p_m, q_n) \leq d(p_n, p_m) + d(q_m, q_n)$ and also $d(p_m, q_m) - d(p_n, q_n) \leq d(p_m, p_n) + d(q_n, q_m)$. More succintly, $|d(p_n, q_n) - d(p_m, q_n)| \leq d(p_n, p_m) + d(q_n, q_m)$.

Given any $\epsilon > 0$, there exist positive integers $N_0, N_1$ such that if $n, m \geq N_0$ then $d(p_n, p_m) < \frac{1}{2} \epsilon$ and if $n, m \geq N_1$ then $d(q_n, q_m) < \frac{1}{2} \epsilon$. Taking $N = \max(N_0, N_1)$, we see (using the above inequality) that if $n, m \geq N$ then $|d(p_n, q_n) - d(p_m, q_n)| \leq d(p_n, p_m) + d(q_n, q_m) < \frac{1}{2} \epsilon + \frac{1}{2} \epsilon = \epsilon$. Therefore $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}$, and since $\mathbb{R}$ is a complete metric space (with the standard metric $d(x, y) = |x - y|$ for $x, y$ in $\mathbb{R}$), the sequence converges.

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5. Chapter 3, p. 82 #24. Let $X$ be a metric space. 
(a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in $X$ equivalent if $\lim_{n \to \infty} d(p_n, q_n) = 0$. Prove that this is an equivalence relation.
(b) Let $X^*$ be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define $\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n)$; by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that $\Delta$ is a distance function in $X^*$.
(c) Prove that the resulting metric space $X^*$ is complete.
(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are $p$; let $P_p$ be the element of $X^*$ which
contains this sequence. Prove that $\Delta(P_p, P_q) = d(p, q)$ for all $p, q \in X$. In other words, the mapping $\varphi$ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of $X$ into $X^*$.

(c) Prove that $\varphi(X)$ is dense in $X^*$, and that $\varphi(X) = X^*$ if $X$ is complete. By (d), we may identify $X$ and $\varphi(X)$ and thus regard $X$ as embedded in the complete metric space $X^*$. We call $X^*$ the completion of $X$.

Solution.

(a) Reflexive. $\lim_{n \to \infty} d(p_n, p_n) = \lim_{n \to \infty} 0 = 0$, so $\{p_n\}$ is equivalent to itself.

Symmetry. $\lim_{n \to \infty} d(p_n, q_n) = \lim_{n \to \infty} d(q_n, p_n)$, so done here.

Transitivity. Suppose $\lim_{n \to \infty} d(p_n, q_n) = 0$ and $\lim_{n \to \infty} d(q_n, h_n) = 0$, where $\{h_n\}$ is also a Cauchy sequence in $X$. Given any $\epsilon > 0$, there exist $N_0, N_1$ positive integers such that if $n \geq N_0$ then $d(p_n, q_n) < \frac{\epsilon}{2}$ and if $n \geq N_1$ then $d(q_n, h_n) < \frac{\epsilon}{2}$, so that taking $N = \max(N_0, N_1)$, we see that if $n \geq N$ then $d(p_n, h_n) \leq d(p_n, q_n) + d(q_n, h_n) < \epsilon$.

So $\lim_{n \to \infty} d(p_n, h_n) = 0$.

(b) Suppose $\{p_n\}$ is equivalent to $\{h_n\}$ and $\{q_n\}$ is equivalent to $\{s_n\}$. Note that $d(p_n, q_n) \leq d(q_n, s_n) + d(s_n, h_n) + d(p_n, h_n)$ and so $d(p_n, q_n) - d(s_n, h_n) \leq d(q_n, s_n) + d(p_n, h_n)$. Likewise, $d(s_n, h_n) - d(p_n, q_n) \leq d(q_n, s_n) + d(p_n, h_n)$. Therefore $|d(p_n, q_n) - d(s_n, h_n)| \leq d(q_n, s_n) + d(p_n, h_n)$.

Let $L_1 = \lim_{n \to \infty} d(p_n, q_n)$ and $L_2 = \lim_{n \to \infty} d(s_n, h_n)$. Take any $\epsilon > 0$. Then there exist positive integers $N_1, N_2, N_3, N_4$ such that if $n \geq N_1$ then $|L_1 - d(p_n, q_n)| < \frac{\epsilon}{4}$, if $n \geq N_2$ then $|L_2 - d(s_n, h_n)| < \frac{\epsilon}{4}$, if $n \geq N_3$ then $d(q_n, s_n) < \frac{\epsilon}{4}$, and if $n \geq N_4$ then $d(p_n, h_n) < \frac{\epsilon}{4}$. Take $n = \max(N_1, N_2, N_3, N_4)$. Then, $|L_1 - L_2| \leq |L_1 - d(p_n, q_n)| + |d(p_n, q_n) - L_2| \leq |L_1 - d(p_n, q_n)| + |d(p_n, q_n) - d(s_n, h_n)| + |d(s_n, h_n) - L_2| \leq |L_1 - d(p_n, q_n)| + |L_2 - d(s_n, h_n)| + d(q_n, s_n) + d(p_n, h_n) < \epsilon$. This being true for all $\epsilon > 0$, $L_1 = L_2$.

Reflexivity and symmetry of $\Delta$ follows trivially. For transitivity, note that if $\{h_n\}$ is another Cauchy sequence then from $d(p_n, q_n) \leq d(p_n, h_n) + d(h_n, q_n)$, we have $\lim_{n \to \infty} d(p_n, q_n) \leq \lim_{n \to \infty} d(p_n, h_n) + \lim_{n \to \infty} d(h_n, q_n)$, or $\Delta(P, Q) \leq \Delta(P, H) + \Delta(H, Q)$, where $H \in X^*$ is represented by $\{h_n\}$.

(c) [Solution in piazza by our Professor Chow]

(d) This simply follows from $\lim_{n \to \infty} d(p_n, q) = d(p, q)$.

(e) Take $P \in X^*$, which is represented by a Cauchy sequence $\{p_n\}$. Take $\epsilon > 0$. Then there exists a positive integer $N$ such that if $n, m \geq N$ then $d(p_n, p_m) < \frac{\epsilon}{2}$. Take $p = p_N$. We claim that $\Delta(\varphi(p_N), P) < \epsilon$.

Let $L = \Delta(\varphi(p_N), P) = \lim_{n \to \infty} d(p_N, p_n)$, which exists by problem #23. There exists $N'$ a positive integer such that if $n \geq N'$ then $|L - d(p_N, p_n)| < \frac{\epsilon}{2}$. Taking $n = \max(N, N')$, we have $L = |L| \leq |L - d(p_N, p_n)| + |d(p_N, p_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So the claim has been proven.

So $\varphi(X)$ is dense in $X^*$.

Suppose now that $X$ is complete. Then there is $p \in X$ such that $\lim_{n \to \infty} p_n = p$. Then $\Delta(\varphi(p), P) = \lim_{n \to \infty} d(p, p_n) = 0$. 
