HW 6. Due date: Monday November 14 at 2 pm

1. (a) Give an example of a sequence \( \{x_n\} \) of integers with \( \limsup_{n \to \infty} x_n = +\infty \) and \( \liminf_{n \to \infty} x_n = -\infty \).

(b) Prove that a sequence \( \{x_n\} \) of real numbers satisfies \( \liminf_{n \to \infty} x_n = +\infty \) if and only if \( \lim_{n \to \infty} x_n = +\infty \).

(c) Prove that a sequence \( \{x_n\} \) of real numbers satisfies \( \liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n + x \) if and only if \( \lim_{n \to \infty} x_n = x \).

2. Let \( (X,d) \) be a metric space with the property that there exists \( p_0 \in X \) such that for every \( n \in \mathbb{Z}^+ \), \( B_n(p_0) = \{ x \in X \mid d(x,p_0) \leq n \} \) is compact. Prove that \( B_r(p) \) is compact for all \( p \in X \) and \( r \in \mathbb{R}^+ \).

3. Prove that every sequence \( \{x_n\} \) of real numbers has a monotonic subsequence.

4. (a) Does the series \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) converge or diverge?

(b) Does the series \( \sum_{n=2}^{\infty} \frac{1}{n(ln n)^2} \) converge or diverge?

Hints: Use substitution and the integral test to evaluate the corresponding improper integrals.

4. (a) Do Chapter 3 Exercise #7 on p. 78.

(b) For which range of \( p \) in \( \mathbb{R}^+ \) can you generalize Exercise #7 to show that \( \sum \frac{\sqrt{n}}{n^p} \) converges under the same hypotheses?

(c) Does \( \sum \frac{\sqrt{n}}{n^{3/2}} \) converge under the same hypotheses?

5. Do Chapter 3 Exercise #8 on p. 79.

6. Do Chapter 3 Exercise #14(a)(b) on p. 80.

7. (a) Give a proof, in your own words (i.e., do not copy verbatim any book), of the Root Test Theorem 3.33(a)(b).

(b) Give a proof, in your own words, of the Ratio Test Theorem 3.34(a)(b).