Complex numbers, complex Euclidean space, Cauchy–Schwarz inequality

The complex numbers $\mathbb{C}$ is the set of expressions of the form $a + bi$ (also written $a + ib$), where $a, b \in \mathbb{R}$ and where $i$ is a symbol whose properties we will explain below. We identify $a + bi$ with the pair $(a, b)$. That is, we a bijection $f : \mathbb{C} \rightarrow \mathbb{R}^2$ (where $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$) defined by $f(a + bi) = (a, b)$. Define addition on $\mathbb{C}$ by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and define multiplication on $\mathbb{C}$ by

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

The way to arrive at this definition of complex multiplication is to assume the rule $i^2 = -1$ in addition to commutativity and distributivity for $+$ and $\cdot$ (exercise: check this). We write $a$ for $a + 0i$ and we write $bi$ for $0 + bi$. In particular, we write $i$ for $0 + 1i$ and $-i$ for $0 + (-1)i$. Then $a + bi$ is really indeed equal to $a + b \cdot i$. (Using the identification $f$, i.e., taking Rudin’s point of view, this says: $(a, b) = (a, 0) + (b, 0) \cdot (0, 1)$, which the reader can easily verify)

The (complex) conjugate of $a + bi$ is $a - bi$ and is written as $\bar{a + bi}$. Theorem 1.31 states the easy facts about the conjugate $\bar{z}$ of $z \in \mathbb{C}$.

The real part of $a + bi$ is $\text{Re} (a + bi) = a$ and the imaginary part of $a + bi$ is $\text{Im} (a + bi) = b$.

**Exercise:** Show that $z\bar{z}$ is real and positive unless $z = 0$, in which case $z\bar{z} = 0$.

The absolute value of $z \in \mathbb{C}$ is $|z| = (z\bar{z})^{1/2}$.

Given $n \in \mathbb{Z}^+$, let $\mathbb{C}^n$ denote the $n$-fold cartesian product of $\mathbb{C}$, i.e.,

$$\mathbb{C}^n = \{(z_1, \ldots, z_n) \mid z_1, \ldots, z_n \in \mathbb{C}\}.$$ 

Define (vector) addition of $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ by

$$z + w = (z_1 + w_1, \ldots, z_n + w_n).$$

The Hermitian inner product of $z, w \in \mathbb{C}^n$ is defined by

$$z \cdot w = z_1\overline{w_1} + \cdots + z_n\overline{w_n} \in \mathbb{C}.$$

The absolute value of $z \in \mathbb{C}^n$ is $|z| = (z \cdot z)^{1/2}$. I.e.,

$$|z| = (z_1\overline{z_1} + \cdots + z_n\overline{z_n})^{1/2} = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}.$$

**Exercise:** (i) Show that for $x, y \in \mathbb{C}^n$,

$$|x + y|^2 = |x|^2 + |y|^2 + x \cdot y + y \cdot x.$$

(ii) Find a similar expression for $|x - y|^2$.

(iii) Show that if $c \in \mathbb{C}$, then $(cx) \cdot y = c(x \cdot y)$ and $x \cdot (cy) = \overline{c} \cdot x \cdot y$.

(iv) Show that $y \cdot x = \overline{x} \cdot y$ and hence $x \cdot y + y \cdot x = 2 \text{Re} (x \cdot y) = 2 \text{Re} (y \cdot x)$.
Theorem 1.35 (Cauchy–Schwarz inequality).

\[ |z \cdot w| \leq |z| |w|. \]

Proof: We compute that (especially check where we use \( x \cdot (cy) = c(x \cdot y) \))

\[
0 \leq |w|^2 |z - (z \cdot w)w|^2 \\
= |w|^4 |z|^2 + |z \cdot w|^2 |w|^2 - (|w|^2 z) \cdot ((z \cdot w)w) - ((z \cdot w)w) \cdot (|w|^2 z) \\
= |w|^4 |z|^2 + |z \cdot w|^2 |w|^2 - |w|^2 |z \cdot w|^2 - |w|^2 |z \cdot w|^2 \\
= |w|^2 (|w|^2 |z|^2 - |z \cdot w|^2).
\]

If \( |w| = 0 \), then the Cauchy–Schwarz inequality is trivial. Otherwise, \( |w| > 0 \). So the above display implies

\[ |w|^2 |z|^2 - |z \cdot w|^2 \geq 0. \]

Exercise. Show that \( |\text{Re} (z \cdot w)| \leq |z| |w| \) and \( |\text{Im} (z \cdot w)| \leq |z| |w| \).

Theorem 1.37. (e):

\[ |w + z| \leq |w| + |z|. \]

And hence (f) (Triangle inequality):

\[ |x - z| \leq |x - y| + |y - z|. \]

Defining the distance between two points by \( \text{dist}(x, y) = |x - y| \). This says

\[ \text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z). \]

Proof. We have

\[
|w + z|^2 = |w|^2 + |z|^2 + 2 \text{Re} (w \cdot z) \\
\leq |w|^2 + |z|^2 + 2 |z| |w| \\
= (|w| + |z|)^2.
\]

The above results apply to real Euclidean space \( \mathbb{R}^n \) (essentially as a special case by taking vectors whose components are real). Here the Hermitian inner product is replaced by the real inner product of \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), defined by

\[ x \cdot y = x_1y_1 + \cdots + x_ny_n \in \mathbb{R}. \]

The Cauchy–Schwarz inequality says for \( x, y \in \mathbb{R}^n \):

\[ |x \cdot y| \leq |x| |y|. \]