Rudin Exercise 2 on p. 43

A complex number $z$ is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + z_n = 0.$$ 

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer $N$ there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

**Hints.** Let $\mathcal{P}$ denote the set of complex polynomials with integer coefficients minus the (trivial) zero polynomial. Define $f : \mathcal{P} \to \mathbb{Z}^+$ by

$$f (a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + z_n) = n + |a_0| + |a_1| + \cdots + |a_n|.$$

Let $\mathcal{P}_N = f^{-1} (N)$. Then $\mathcal{P} = \bigcup_{N \in \mathbb{Z}^+} \mathcal{P}_N$ (is a disjoint union). Let $\mathcal{A}$ denote the set of algebraic numbers. For $N \in \mathbb{Z}^+$, let $\mathcal{A}_N$ denote the set of complex zeroes of polynomials in $\mathcal{P}_N$, i.e.,

$$\mathcal{A}_N = \{ z \in \mathbb{C} | P(z) = 0 \text{ for some } P \in \mathcal{P}_N \}.$$

Then since $\mathcal{P} = \bigcup_{N \in \mathbb{Z}^+} \mathcal{P}_N$ we have

$$\mathcal{A} = \bigcup_{N \in \mathbb{Z}^+} \mathcal{A}_N.$$

Now each $\mathcal{P}_N$ is a finite set (see Rudin’s hint above). Moreover, each $P \in \mathcal{P}$ has only a finite number of zeroes (by the Fundamental Theorem of Algebra).

Note that, as I mentioned briefly in class, Theorem 2.12 generalizes easily to the statement:

*Let $\{E_n\}, n = 1, 2, 3, \ldots$, be a sequence of at most countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n.$$*

Then $S$ is at most countable.

Finally, give a very simple reason why the set $\mathcal{A}$ is infinite. (Consider linear, i.e., degree one, polynomials.)