**Heine–Borel Theorem**

**Theorem 2.41.** Let $E \subset \mathbb{R}^k$. The following are equivalent:

(a) $E$ is closed and bounded.

(b) $E$ is compact.

(c) Every infinite subset of $E$ has a limit point in $E$.

**Proof.** We shall show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Let $E \subset \mathbb{R}^k$ be a closed and bounded subset. Since $E$ is bounded, there exists a $k$-cell $K$ such that $E \subset K$. Since $E$ is a closed subset of a compact set, $E$ is compact.

(b) $\Rightarrow$ (c) [Theorem 2.37]. Let $E$ be a compact set and let $S$ be an infinite subset of $E$. Suppose that $S$ has no limit points in $E$. Then for each $p \in E$ there exists $r(p) > 0$ such that $B_{r(p)}(p) \cap S \subset \{p\}$. Now $\{B_{r(p)}(p)\}_{p \in E}$ is an open cover of the compact set $E$. Since $E$ is compact, there exists a finite subcover $\{B_{r(p_i)}(p_i)\}_{i=1}^k$ for $E$. Since $S \subset E$, we thus have

$$S \subset \bigcup_{i=1}^k (B_{r(p_i)}(p_i) \cap S) \subset \bigcup_{i=1}^k \{p_i\},$$

which implies that $S$ is finite, a contradiction.

(c) $\Rightarrow$ (a). Suppose that $E \subset \mathbb{R}^k$ is such that every infinite subset of $E$ has a limit point in $E$.

(i). $E$ is bounded. Suppose that $E$ is not bounded. Then for each $n \in \mathbb{Z}^+$ there exists $x_n \in E$ such that $|x_n| > n$. It is easy to see that the infinite set $S = \bigcup_{i=1}^{\infty} \{x_n\} \subset E$ does not have a limit point (exercise: prove this). This contradiction implies that $E$ is bounded.

(ii). $E$ is closed. Suppose that $E$ is not closed. Then there exists $x_0 \in E^c$ that is a limit point of $E$. Hence for each $n \in \mathbb{Z}^+$ there exists $x_n \in E$ such that $|x_n - x_0| < \frac{1}{n}$. Let $S = \bigcup_{n \in \mathbb{Z}^+} \{x_n\}$ (exercise: show that $S$ is infinite).

Suppose that $S$ has a limit point $y_0 \in E$. Let $n_0 \in \mathbb{Z}^+$ be such that $|y_0 - x_0| > \frac{2}{n_0}$. Since $y_0$ is a limit point of $S$, there exist $n_1 \geq n_0$ such that $|x_{n_1} - y_0| < \frac{1}{n_0}$. We conclude that

$$|y_0 - x_0| \leq |x_{n_1} - y_0| + |x_{n_1} - x_0| < \frac{1}{n_0} + \frac{1}{n_1} \leq \frac{2}{n_0}.$$

This contradicts $|y_0 - x_0| > \frac{2}{n_0}$. Hence $S$ has no limit points in $E$. Since $S$ is infinite, this contradicts the assumption that every infinite subset of $E$ has a limit point in $E$. Therefore $E$ is closed. □

**Corollary 1 (Theorem 2.42 Weierstrass).** Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

**Proof.** Let $E$ be a bounded infinite subset of $\mathbb{R}^k$. Then there exists a $k$-cell $K$ such that $E \subset K$. Since $K$ is compact and since $E \subset K$ is infinite, by Theorem 2.41 $E$ has a limit point in $K$. □
Corollary 2. Any bounded sequence \( \{x_n\}_{n \in \mathbb{Z}^+} \) in \( \mathbb{R}^k \) has a convergent subsequence.

Proof. Let \( S = \bigcup_{n \in \mathbb{Z}^+} \{x_n\} \).

Case 1. \( S \) is finite. Then \( S = \{y_1, \ldots, y_k\} \). Let \( Z_i = \{n \mid x_n = y_i\} \) for \( i = 1, \ldots, k \). Then \( \bigcup_{i=1}^k Z_i = \mathbb{Z} \). Hence there exists \( i \) such that \( Z_i \) is infinite. Thus \( Z_i = \bigcup_{j \in \mathbb{Z}^+} \{n_j\} \), where \( n_1 < n_2 < \cdots \). We have \( x_n_j = y_i \) for all \( j \in \mathbb{Z}^+ \). So of course \( \lim_{j \to \infty} x_{n_j} = y_i \).

Case 2. \( S \) is infinite. Since the sequence \( \{x_n\}_{n \in \mathbb{Z}^+} \) is bounded, so is the set \( S \). By Theorem 2.42, \( S \) has a limit point \( x_0 \) in \( \mathbb{R}^k \). Let \( n_1 = 1 \). By induction suppose that we have defined \( n_1 < n_2 < \cdots < n_k \), where \( k \geq 1 \). Let \( T = \bigcup_{n > n_k} \{x_n\} \). Then \( x_0 \) is a limit point of \( T \) (exercise: prove this). Hence there exists an integer \( n_{k+1} > n_k \) such that \( |x_{n_{k+1}} - x_0| < \frac{1}{k+1} \). Then \( \{x_{n_k}\}_{k \in \mathbb{Z}^+} \) is a subsequence of \( \{x_n\}_{n \in \mathbb{Z}^+} \) satisfying \( |x_{n_k} - x_0| < \frac{1}{k} \) for all \( k \geq 2 \). Therefore \( \lim_{k \to \infty} x_{n_k} = x_0 \). \( \square \)