Homework assignment 8, due Wednesday, 3/11.

p. 79 #10, p. 98 #2, p. 99 #10, #12, #14, #20. $\neq$ means defined to be equal to.

p. 79 #10: There exist positive integers $n_1 < n_2 < n_3 < \cdots$ such that $a_{n_k} \neq 0$ for $k \geq 1$. Therefore $|a_{n_k}| \geq 1$. This implies

$$\limsup_{n \to \infty} \sqrt[n]{|a_n z^n|} \geq \limsup_{k \to \infty} \sqrt[|a_{n_k}|]{|z|} \geq |z|.$$ 

Hence, if $|z| > 1$, the series $\sum a_n z^n$ diverges. This implies that its radius of convergence is at most 1.

p. 98 #2: Part 1 (1st version): $f(E) \subset \overline{f(E)}$.

(i) $x \in E$. Then $f(x) \in f(E) \subset \overline{f(E)}$.

(ii) $x \in E'$. Suppose that $f(x) \notin f(E)$. Let $\varepsilon > 0$. Since $f$ is continuous at $x$, there exists $\delta > 0$ such that $f(z) \in N_\varepsilon(f(x))$ for all $z \in N_\delta(x)$. Since $x \in E'$, there exists $x_1 \in N_\delta(f(x))$. Then $f(x_1) \in N_\varepsilon(f(x))$. Since $x_1 \in E$, $f(x_1) \in f(E)$. Since $f(x) \notin f(E)$, we have $f(x_1) \neq f(x)$. Summarizing, we have proved: For any $\varepsilon > 0$ there exists $f(x_1) \in f(E)$ such that $f(x_1) \in N_\varepsilon(f(x)) - \{f(x)\}$. This proves that $f(x)$ is a limit point of $f(E)$. In particular, $f(x) \in \overline{f(E)}$.

We have proved that if $x \in E$, then $f(x) \in \overline{f(E)}$. We conclude that $f(E) \subset \overline{f(E)}$.

Part 1 (2nd version): $f(E) \subset \overline{f(E)}$.

(i) $x \in E$. Then $f(x) \in f(E) \subset \overline{f(E)}$. (Same as above.)

(ii) $x \in E'$. Then there exists a sequence $\{x_n\}$ with $x_n \in E$ and $x_n \to x$ (by Theorem 3.2(d)). Since $f$ is continuous (at $x$), we have $f(x_n) \to f(x)$ (this is a consequence of Theorem 4.2).

Case (a): There exists $n \in \mathbb{N}$ such that $f(x_n) = f(x)$. Since $x_n \in E$, we have $f(x_n) \in f(E)$. Since $f(x_n) = f(x)$, we conclude that $f(x) \in f(E)$.

Case (b): $f(x_n) \neq f(x)$ for all $n \in \mathbb{N}$. Then, since $f(x_n) \in f(E)$ and $f(x_n) \to f(x)$, we conclude that $f(x) \in f(E)$.

In either case (a) or (b), we have $f(x) \in \overline{f(E)}$.

Part 2: $f(E)$ can be a proper subset of $\overline{f(E)}$.

Let $X = \mathbb{N}$ and let $Y = \mathbb{R}$. Define $f: \mathbb{N} \to \mathbb{Q} \subset \mathbb{R}$ to be any bijection from $\mathbb{N}$ to $\mathbb{Q}$, which exists since $\mathbb{Q}$ is countable. Since every point of $\mathbb{N}$ is an isolated point, any function $f: \mathbb{N} \to \mathbb{R}$ is continuous. We have $f(\mathbb{N}) = f(\mathbb{N}) = \mathbb{Q}$, whereas $f(\overline{\mathbb{N}}) = \overline{\mathbb{Q}} = \mathbb{R}$. Therefore $f(\mathbb{N})$ is a proper subset of $f(\overline{\mathbb{N}})$. $\square$

99 #10: Let $f: X \to Y$ be continuous and $X$ be compact. Suppose that $f$ is not uniformly continuous. Then there exists $\varepsilon > 0$ such that there does not exist $\delta > 0$ such that $d_Y(f(p), f(q)) < \varepsilon$ whenever $d_X(p, q) < \delta$. Hence by applying this statement to $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$, there exist $p_n, q_n \in X$ such that $d_X(p_n, q_n) < \frac{1}{n}$ and $d_Y(f(p_n), f(q_n)) \geq \varepsilon$. This implies $p_n \neq q_n$. Of course, $d_X(p_n, q_n) \to 0$.

Since $\{p_n\}$ is a sequence in the compact metric space $X$, by Theorem 3.6(a) there exists a subsequence $\{p_{n_k}\}$ and $p \in X$ such that $p_{n_k} \to p$. Since $d_X(p_{n_k}, q_{n_k}) < \frac{1}{n_k}$, we also have $q_{n_k} \to p$. On the other hand, since $f$ is continuous at $p$, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \frac{\varepsilon}{2}$ for all $x$ such that $d_X(x, p) < \delta$. Since $p_{n_k} \to p$ and $q_{n_k} \to p$, there exists $n_k \in \mathbb{N}$ such that $d_X(p_{n_k}, p) < \delta$ and $d_X(q_{n_k}, p) < \delta$. Then $d_Y(f(p_{n_k}), f(p)) < \frac{\varepsilon}{2}$ and $d_Y(f(q_{n_k}), f(p)) < \frac{\varepsilon}{2}$. We conclude that

$$d_Y(f(p_{n_k}), f(q_{n_k})) \leq d_Y(f(p_{n_k}), f(p)) + d_Y(f(q_{n_k}), f(p)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which contradicts $d_Y(f(p_{n_k}), f(q_{n_k})) \geq \varepsilon$. 

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#12: Let \( f : X \to Y \) and \( g : Y \to Z \) be uniformly continuous functions. Then \( g \circ f : X \to Z \) is uniformly continuous.

**Proof.** Since \( g \) is uniformly continuous, for every \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that \( d_Z (g(r), g(s)) < \varepsilon \) whenever \( d_Y(r, s) < \eta \). Since \( f \) is uniformly continuous, there exists \( \delta > 0 \) such that \( d_Y(f(p), f(q)) < \eta \) whenever \( d_X(p, q) < \delta \). Therefore, if \( d_X(p, q) < \delta \), then since \( d_Y(f(p), f(q)) < \eta \), we have

\[
d_Z((g \circ f)(p), (g \circ f)(q)) = d_Z(g(f(p)), g(f(q))) < \varepsilon.
\]

We have proved that \( g \circ f \) is uniformly continuous.

#14: Let \( f : I \to I \) be a continuous mapping, where \( I = [0, 1] \). Then there exists \( x \in I \) such that \( f(x) = x \).

**Proof.** If \( f(0) = 0 \) or \( f(1) = 1 \), then we are done. So we may assume that \( f(0) > 0 \) and \( f(1) < 1 \). Define \( g : I \to \mathbb{R} \) by \( g(x) = x - f(x) \). Then \( g(0) = -f(0) < 0 \) and \( g(1) = 1 - f(1) > 0 \). Since \( g \) is continuous on \([0, 1]\) and \( g(0) < 0 < g(1) \), there exists \( x \in (0, 1) \) such that \( g(x) = 0 \). That is, \( f(x) = x \).

#20: \( E \subseteq X \) is nonempty. \( \rho_E(x) \doteq \inf_{z \in E} d(x, z) \).

(a) \( \rho_E(x) = 0 \) if and only if \( x \in E \).

(b) \( \rho_E \) is uniformly continuous on \( X \).

\[
|\rho_E(x) - \rho_E(y)| \leq d(x, y) \quad \text{for } x, y \in X.
\]

Hint: \( \rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z) \), so that \( \rho_E(x) \leq d(x, y) + \rho_E(y) \).

**Proof of (a).** \( \Rightarrow \) Suppose \( \inf_{z \in E} d(x, z) = 0 \). Let \( \varepsilon > 0 \). Then there exists \( z \in E \) such that \( d(x, z) < \varepsilon \). Thus \( x \in E \).

\( \Leftarrow \) Suppose \( x \in E \). If \( x \in E \), then \( \inf_{z \in E} d(x, z) = d(x, x) = 0 \), so \( \rho_E(x) = 0 \). So we may assume that \( x \in E' \). Let \( \varepsilon > 0 \). Since \( x \) is a limit point of \( E \), there exists \( z \in E \) such that \( d(x, z) < \varepsilon \). Thus \( \rho_E(x) < \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, \( \rho_E(x) = 0 \).

**Proof of (b).** Let \( x, y \in X \). Then for any \( z \in E \) we have

\[
\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)
\]

by the definition of \( \rho_E \) and the triangle inequality. Therefore

\[
\rho_E(x) \leq d(x, y) + \inf_{z \in E} d(y, z) = d(x, y) + \rho_E(y).
\]

Thus

\[
\rho_E(x) - \rho_E(y) \leq d(x, y).
\]

By switching \( x \) and \( y \) in the above argument, we obtain

\[
\rho_E(y) - \rho_E(x) \leq d(x, y).
\]

The result follows.