Quiz #1  Math 20D  January 16  SOLUTIONS

1. (§2.1, #30. 10 points) Find the value of $y_0$ for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \quad y(0) = y_0$$

remains finite as $t \to \infty$.

ANSWER:

(1) $p(t) = -1$

(2) $\mu(t) = e^{\int p(t)\,dt} = e^{-t}$ (some students will be able to figure this out directly, that is OK)

(3) Thus

$$y(t) = \frac{1}{\mu(t)} \left( \int g(t)\mu(t)\,dt + C \right)$$

$$= e^t \left( \int e^{-t} (1 + 3\sin t)\,dt + C \right)$$

$$= e^t \left( -e^{-t} + 3 \int e^{-t} \sin t\,dt + C \right)$$

(Give say 7 points if students get to here correctly.)

(4) Compute $\int e^{-t} \sin t\,dt$ by integration by parts twice. In particular: Let

$$u = \sin t, \quad dv = e^{-t}\,dt,$$

$$du = \cos t\,dt, \quad v = -e^{-t}.$$

Then

$$\int e^{-t} \sin t\,dt = -e^{-t} \sin t + \int e^{-t} \cos t\,dt.$$

Now let

$$u = \cos t, \quad dv = e^{-t}\,dt,$$

$$du = -\sin t\,dt, \quad v = -e^{-t}.$$

Then

$$\int e^{-t} \cos t\,dt = -e^{-t} \cos t - \int e^{-t} \sin t\,dt.$$

Hence

$$\int e^{-t} \sin t\,dt = -e^{-t} \sin t - e^{-t} \cos t - \int e^{-t} \sin t\,dt,$$

so that

$$\int e^{-t} \sin t\,dt = -\frac{1}{2} e^{-t} (\sin t + \cos t).$$

So

$$y(t) = e^t \left( -e^{-t} + 3 \int e^{-t} \sin t\,dt + C \right)$$

$$= -1 - \frac{3}{2} (\sin t + \cos t) + Ce^t.$$

Now $y(0) = y_0$, so that

$$y_0 = y(0) = -1 - \frac{3}{2} + C = -\frac{5}{2} + C.$$

Therefore

$$y(t) = -1 - \frac{3}{2} (\sin t + \cos t) + \left( y_0 + \frac{5}{2} \right) e^t.$$
The only way this can be bound is if

\[ y_0 + \frac{5}{2} = 0, \]

that is

\[ y_0 = -\frac{5}{2}. \]

2. (§2.2, #24. 10 points) Solve the initial value problem

\[ y' = \frac{2 - e^x}{3 + 2y}, \quad y(0) = 0 \]

and determine where the solution attains its maximum value.

**ANSWER:** This is a separable equation:

\[ (3 + 2y) \, dy = (2 - e^x) \, dx, \]

so (integrate)

\[ 3y + y^2 = 2x - e^x + C. \]

Since \( y(0) = 0 \), we have

\[ 0 = -1 + C, \]

so that \( C = 1 \). We get

\[ 2x - e^x + 1 = 3y + y^2 = \left(y + \frac{3}{2}\right)^2 - \frac{9}{4}, \]

so that

\[ \left(y + \frac{3}{2}\right)^2 = 2x - e^x + \frac{13}{4}. \]

We take the positive square root to get:

\[ y = -\frac{3}{2} + \sqrt{2x - e^x + \frac{13}{4}}. \]

The \( x \) value where \( y \) attains its maximum is where

\[ 2x - e^x + \frac{13}{4} \]

attains its maximum. Setting the derivative to be zero, we get

\[ 0 = \frac{d}{dx} \left(2x - e^x + \frac{13}{4}\right) = 2 - e^x. \]

So it is where \( e^x = 2 \), that is,

\[ x = \ln 2. \]