Proposition 23.1.2. Every integer greater than 1 can be written as the product of prime numbers.
In other words, every integer $\geq 2$ is either a prime or is the product of 2 or more primes.
For example, the first 9 such numbers are: 2, 3, 4 = 2 · 2, 5, 6 = 2 · 3, 7, 8 = 2 · 2 · 2, 9 = 3 · 3, 10 = 2 · 5.

Proof of Proposition 23.1.2 (by strong induction). Recall that in strong induction we need to prove the base case and the following:
If $P(1), P(2), \ldots, P(k)$ are true for some $k \in \mathbb{Z}^+$, then $P(k + 1)$ is true.

(1) Base case: 2 is a prime, so it is the product of a single prime.

(2) Strong inductive step: Suppose that for some $k \geq 2$ each integer $n$ satisfying $2 \leq n \leq k$
may be written as a product of primes. We need to prove that $k + 1$ is a product of primes.

Case (a): Suppose $k + 1$ is a prime. Then we are done.
Case (b): Suppose $k + 1$ is a not prime. Then there exist integers $a, b$ with $2 \leq a, b \leq k$ such that
$$k + 1 = a \cdot b.$$ 

By the strong inductive hypothesis, both $a$ and $b$ are the product of primes. Thus $k + 1 = a \cdot b$ is the
product of primes.

(3) We are done by strong induction.

Theorem 23.5.1. There are infinitely many primes.

Proof (by contradiction). Suppose that the set of all primes is finite. Then we may write this set
as
$$P = \{p_1, p_2, \ldots, p_n\},$$
where $n$ is the cardinality of this set of all primes. By Proposition 23.1.2, for any integer $m > 1$ there
exists $i$ such that $p_i$ divides $m$. On the other hand, for each $i$, $p_i$ does not divide the integer
$$m = p_1 p_2 \cdots p_n + 1$$
since $m \equiv 1 \mod p_i$ and $p_i > 1$. Since $m > 1$, we obtain a contradiction.

Remark. In other words, suppose $p_i$ divides $p_1 p_2 \cdots p_n + 1$ for some $1 \leq i \leq n$. Since $p_i$ divides
$p_1 p_2 \cdots p_n$, this implies that $p_i$ divides 1, a contradiction.