Change of variables—Spherical coordinates (§6.2)

Consider the function (we’ll call this the ‘spherical coordinates to cartesian coordinates map’)

\[ T : \mathbb{R}^3 \to \mathbb{R}^3 \]

defined by

\[ T(\rho, \theta, \phi) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi). \]

Since

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta, \\
z &= \rho \cos \phi,
\end{align*}
\]

we calculate that the Jacobian determinant of \( T \) is

\[
\frac{\partial (x, y, z)}{\partial (\rho, \theta, \phi)} = \begin{vmatrix}
x & y & z \\
\rho \sin \phi \cos \theta & \rho \sin \phi \sin \theta & \rho \cos \phi \\
\rho \sin \phi \cos \theta & \rho \cos \phi \\
\rho \cos \phi
\end{vmatrix} = \rho^2 \sin \phi,
\]

where the last equality arises after some simplification (see p. 388 for details).

The basic change of variables formula is: If \( T \) is a one-to-one continuously differentiable map on \( W \subset \mathbb{R}^3 \), then

\[
\iiint_{T(W)} f(x, y, z) 
\]

\[
= \iiint_{W} \left| \frac{\partial (x, y, z)}{\partial (\rho, \theta, \phi)} \right| 
\]

\[
= \iiint_{W} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \, \rho^2 \sin \phi \, d\rho d\theta d\phi.
\]