Paths

A path \( \vec{c} \) is simply a function from an interval in \( \mathbb{R} \) into some Euclidean space such as \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). The set of values of \( \vec{c} \) is called the curve that \( \vec{c} \) traces out.

Take for example the path

\[
\vec{c} : [0, 6\pi] \to \mathbb{R}^2
\]

defined by

\[
\vec{c}(t) = (4 \cos t, 4 \sin t).
\]

This traces out the curve \( C \) which is the circle of radius 4 centered at \((0, 0)\), going around counterclockwise 3 times (since \(6\pi = 3 \cdot 2\pi\)). Note that

\[
C = \{(x, y) : x^2 + y^2 = 16\}.
\]

For some reason, usually we don’t think much about the graph of a path. But, in any case, for the above path it is

\[
\text{graph } \vec{c} = \{(t, 4 \cos t, 4 \sin t) \mid 0 \leq t \leq 6\pi\},
\]

which is a subset of \( \mathbb{R}^3 \). This is a spiral around the \( x \)-axis.

Tangent vector to a path

Let \( \vec{c} : (a, b) \to \mathbb{R}^n \) be a path and let \( a < t < b \). We say that \( \vec{c} \) is differentiable at \( t \) if

\[
\vec{c}'(t) = \lim_{h \to 0} \frac{\vec{c}(t + h) - \vec{c}(t)}{h}
\]

exists. We call \( \vec{c}'(t) \) the velocity of \( \vec{c} \) or the tangent vector of \( \vec{c} \). The speed of \( \vec{c} \) is \( s = \|\vec{c}'(t)\| \).

Tangent line to a path

Suppose that \( a < t_0 < b \) is such that \( \|\vec{c}'(t_0)\| \neq 0 \). The tangent vector \( \vec{c}'(t_0) \), which begins at the point \( \vec{c}(t_0) \), determines a line, called the tangent line to \( \vec{c} \). An equation for this line is easy seen to be:

\[
\vec{l}(t) = \vec{c}(t_0) + (t - t_0) \vec{c}'(t_0).
\]

Note that this form is rigged so that \( \vec{l}(t_0) = \vec{c}(t_0) \). Note that \( \vec{l}(t) \) is actually the linear approximation of \( \vec{c}(t) \).