To clarify the logic in the statement of the Induction Principle, we state things more formally.

**Axiom 5.1.1. Induction Principle.** Let \( P(n) \), \( n \in \mathbb{Z}^+ \), be a sequence of statements. If
\[
P(1) \text{ and } (\forall k \in \mathbb{Z}^+, P(k) \Rightarrow P(k + 1)),
\]
then \( \forall n \in \mathbb{Z}^+, P(n) \).

**Axiom 5.4.1. Strong Induction Principle.** Let \( P(n) \), \( n \in \mathbb{Z}^+ \), be a sequence of statements. If
\[
P(1) \text{ and } (\forall k \in \mathbb{Z}^+, (P(1), \ldots, P(k)) \Rightarrow P(k + 1)),
\]
then \( \forall n \in \mathbb{Z}^+, P(n) \).

In class I stated that the Strong Induction Principle implies the Induction Principle.

One thing that is confusing is in each of these statements is that there is an implication inside the hypothesis of an implication. Think about how confusing a dream inside a dream is, such as in the movie *Inception*.

Let \( P \) be \( P(1) \).
Let \( Q \) be \( \forall k \in \mathbb{Z}^+, P(k) \Rightarrow P(k + 1) \).
Let \( Q' \) be \( \forall k \in \mathbb{Z}^+, (P(1), \ldots, P(k)) \Rightarrow P(k + 1) \).
Let \( R \) be \( \forall n \in \mathbb{Z}^+, P(n) \).

**Induction Principle:** \( (P \text{ and } Q) \Rightarrow R \).

**Strong Induction Principle:** \( (P \text{ and } Q') \Rightarrow R \).

The key is:

**Observation.** \( Q \) implies \( Q' \).

**Proof.** Suppose \( \forall k \in \mathbb{Z}^+, P(k) \Rightarrow P(k + 1) \). Then, since \((P(1), \ldots, P(k)) \Rightarrow P(k)\), we have \((P(1), \ldots, P(k)) \Rightarrow P(k + 1) \).

We can now clarify the following statement, which was proved in class.

**Theorem 1.** If the Strong Induction Principle is true, then the Induction Principle is true.

**Proof.** Suppose the Strong Induction Principle is true. Then \((P \text{ and } Q') \Rightarrow R\) is true.

Suppose \( P \) and \( Q \). Since \( Q \Rightarrow Q' \), we have \( P \) and \( Q' \). Hence \( R \) (by \((P \text{ and } Q') \Rightarrow R\)). We have proved that \((P \text{ and } Q) \Rightarrow R\). That is, the Induction Principle is true. □

The converse is also true. Therefore the **Strong Induction Principle** and the **Induction Principle** are equivalent.

**Theorem 2.** If the Induction Principle is true, then the Strong Induction Principle is true.

**Proof.** Suppose the Induction Principle is true.

Suppose \( P \) and \( Q' \). Then \( P(1) \) and \( \forall k \in \mathbb{Z}^+, (P(1), \ldots, P(k)) \Rightarrow P(k + 1) \). Let \( O(k) \) be the statement that \( P(1), \ldots, P(k) \) are true. Then \( O(1) \) is true since \( O(k) \) is the same as \( P(k) \).

Suppose \( k \in \mathbb{Z}^+ \) is such that \( O(k) \) is true. Then \( P(k + 1) \) is true since \( O(k) \Rightarrow P(k + 1) \). Thus \( P(1), \ldots, P(k), P(k + 1) \) are true, that is, \( O(k + 1) \) is true. We have proved that \( O(k) \Rightarrow O(k + 1) \).

By the Induction Principle, we conclude that \( O(n) \) is true for all \( n \in \mathbb{Z}^+ \). This immediately implies that \( P(n) \) is true for all \( n \in \mathbb{Z}^+ \). We conclude that the Strong Induction Principle is true. □

**Remark:** For Induction, the hypothesis in \( \forall k \in \mathbb{Z}^+, P(k) \Rightarrow P(k + 1) \) is weaker than the hypothesis for Strong Induction in \( \forall k \in \mathbb{Z}^+, (P(1), \ldots, P(k)) \Rightarrow P(k + 1) \). So the implication \( \forall k \in \mathbb{Z}^+, P(k) \Rightarrow P(k + 1) \) is stronger than (i.e., implies) the implication \( \forall k \in \mathbb{Z}^+, (P(1), \ldots, P(k)) \Rightarrow P(k + 1) \).

Since the stronger statement \( \forall k \in \mathbb{Z}^+, P(k) \Rightarrow P(k + 1) \) is in the hypothesis of Induction, whereas the weaker statement \( \forall k \in \mathbb{Z}^+, (P(1), \ldots, P(k)) \Rightarrow P(k + 1) \) is in the hypothesis of Strong Induction, whereas they have the same conclusion \( \forall n \in \mathbb{Z}^+, P(n) \), we have that Induction is a weaker statement than Strong Induction. That is, Strong Induction implies Induction.