Math 109. Summer Session I, 2016. Midterm 2. (10 points each question; 60 points total)

Instructions:
(1) On blue book: Name, PID, and Section number.
(2) Write clearly and give a reasonable amount of explanation.
(3) \( \mathbb{Z} \) is the set of integers. \( \mathbb{Z}^+ \) is the set of positive integers.

1. **Inclusion-Exclusion Principle.** Of the 131 students in our class,
   i. 64 caught Arbok, 56 caught Beedrill, and 65 caught Caterpie,
   ii. 26 caught Arbok and Beedrill, 28 caught Arbok and Caterpie, and 23 caught Beedrill and Caterpie,
   iii. 15 caught no Pokémon.

   (a) State the Inclusion-Exclusion Principle for 3 sets \( A, B \) and \( C \).
   (b) How many students in our class caught all three Pokémon?

2. **Euclidean Algorithm.**
   (a) Use the Euclidean Algorithm to find \( g = \gcd (1469, 1027) \).
   (b) Reverse the Euclidean Algorithm to find integers \( m \) and \( n \) such that \( g = 1469m + 1027n \).

3. **Linear Diophantine Equation.** For this problem the universal set is \( \mathbb{Z} \).
   (a) Find a (particular) solution \((m_0, n_0)\) to the equation \(144m + 84n = 24\).
   (b) Let \((m_1, n_1)\) be any solution to \(144m + 84n = 24\). Prove (directly) that 7 divides \(m_1 - m_0\)
   and that 12 divides \(n_1 - n_0\).
   (c) Prove that there exists \(k \in \mathbb{Z}\) such that \(m_1 - m_0 = 7k\) and \(n_1 - n_0 = -12k\).

4. **Uniqueness Statement in the Division Theorem.** Let \(a \in \mathbb{Z}\) and \(b \in \mathbb{Z}^+\). Suppose that \(q_1, q_2, r_1, r_2\) are integers such that
   \[
   a = bq_1 + r_1, \quad 0 < r_1 \leq b, \\
   a = bq_2 + r_2, \quad 0 < r_2 \leq b. 
   \]
   Prove (directly) that \(q_1 = q_2\) and \(r_1 = r_2\). (This problem is related to testing your understanding
   of the proof of the uniqueness part of the division theorem.)
5. **Cartesian product and functions.** Let \( S = \{a, b\} \) and let \( Y \) be a set. Define

\[
f : Y^2 \to \text{Fun}(S, Y)
\]

so that \( f(x_1, y_1) \in \text{Fun}(S, Y) \), that is \( f(x_1, y_1) : S \to Y \), is given by

\[
f(x_1, y_1)(a) = x_1, \\
f(x_1, y_1)(b) = y_1.
\]

(a) Suppose that \((x_1, y_1)\) and \((x_2, y_2)\) are such that \( f(x_1, y_1) = f(x_2, y_2) \) (as elements of \( \text{Fun}(S, Y) \)). Prove that \((x_1, y_1) = (x_2, y_2)\).

(b) Let \( g \in \text{Fun}(S, Y) \), that is \( g : S \to Y \). Prove that there exists \((x, y) \in Y\) such that \( f(x, y) = g \).

(This problem tests your understanding of proving that \( f \) is a bijection.)

6. Define the remainder function \( r_7 : \mathbb{Z} \to R_7 = \{ r \in \mathbb{Z} \mid 0 \leq r < 7 \} \) by \( r_7(a) = r \), where \( r \in R_7 \) is the unique integer such that there exists \( q \in \mathbb{Z}, a = 7q + r \) (provided by the division theorem).

(a) Prove (directly) that if \( r_7(a_1) = r_7(a_2) \), then there exists \( q \in \mathbb{Z}, a_1 - a_2 = 7q \).

(b) Given the following table:

\[
\begin{align*}
(7q)^2 &= 7(7q^2) + 0, \\
(7q + 1)^2 &= 7(7q^2 + 2q) + 1, \\
(7q + 2)^2 &= 7(7q^2 + 4q) + 4, \\
(7q + 3)^2 &= 7(7q^2 + 6q + 1) + 2, \\
(7q + 4)^2 &= 7(7q^2 + 8q + 2) + 2, \\
(7q + 5)^2 &= 7(7q^2 + 10q + 3) + 4, \\
(7q + 6)^2 &= 7(7q^2 + 12q + 5) + 1,
\end{align*}
\]

explain why if \( a \in \mathbb{Z} \), then \( r_7(a^2) \in \{0, 1, 2, 4\} \).

(c) Prove that 777, 777, 775 is not the square of any integer.