Problem HW2:#2: Prove that for any real numbers $a, b, c, d$,

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2).$$

Hint: a backwards proof (as in section 3.2) may be the easiest.

Proof. Working backwards (first expanding each square):

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2) \iff (ac)^2 + 2ac \cdot bd + (bd)^2 \leq a^2 c^2 + a^2 d^2 + b^2 c^2 + b^2 d^2$$

$$\iff 2ac \cdot bd \leq a^2 d^2 + b^2 c^2 \text{ (cancelling terms)}$$

$$\iff 0 \leq (ad - bc)^2 \text{ (completing the square)}.$$

Since the last equality is evidently true (the square of a real number is always nonnegative), this completes the proof.

Problem HW2:#3: Prove by induction on $n$ that, for all positive integers $n$,

$$1^2 + 3^2 + \cdots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}.$$

(1) Base case. $1^2 = 1 = \frac{3}{3} = \frac{1(2(1-1)(2(1+1))}{3}$.

(2) Inductive step. Suppose $k \in \mathbb{Z}_+$ is such that $1^2 + 3^2 + \cdots + (2k - 1)^2 = \frac{k(2k - 1)(2k + 1)}{3}$. Then

$$1^2 + 3^2 + \cdots + (2(k + 1) - 1)^2 = \left[1^2 + 3^2 + \cdots + (2k - 1)^2\right] + (2(k + 1) - 1)^2 \text{ (relate } k + 1 \text{ case to } k \text{ case)}$$

$$= \frac{k(2k - 1)(2k + 1)}{3} + (2k + 1)^2 \text{ (induction hypothesis)}$$

$$= \frac{k(2k - 1)(2k + 1) + 3(2k + 1)^2}{3} \text{ (common denominator)}$$

$$= \frac{(2k + 1)(k(2k - 1) + 3(2k + 1))}{3} \text{ (factor out } 2k + 2\text{)}$$

$$= \frac{(2k + 1)2k^2 + 5k + 3}{3} \text{ (expand and combine)}$$

$$= \frac{(2k + 1)(k + 1)(2k + 3)}{3} \text{ (factor } 2k^2 + 5k + 3\text{)}$$

$$= \frac{(k + 1)(2(k + 1) - 1)(2(k + 1) + 1)}{3} \text{ (rewrite to verify } k + 1 \text{ case)}.$$

(3) We are done by induction.

Problem HW3:#4: Disprove the statement: If $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ satisfy $a^2 + 15 = b^2 + 15$, then $a = b$.

Disproof. The key observation is that for any integer $c$, we have $(-c)^2 = c^2$. So, for example, the choices $a = -1$ and $b = 1$ satisfy

$$a^2 + 15 = 16 = b^2 + 15, \text{ but } a \neq b.$$

Remark. We are to disprove the statement: for each $a, b \in \mathbb{Z}$ satisfying $a^2 + 15 = b^2 + 15$, we have $a = b$. So we just have to find one pair of choices of $a$ and $b$ such that $a^2 + 15 = b^2 + 15$ is true and $a = b$ is false. We did this above.

Problem HW3:#6: (a) Prove that for every $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $y + x^2y = 1$. 

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Proof. Just some algebra. Let $x \in \mathbb{R}$. Then working backwards we have

$$y + x^2y = 1 \iff y(1 + x^2) = 1 \iff y = \frac{1}{1 + x^2}.$$ 

So we let $y = \frac{1}{1 + x^2}$. By the implications, this choice of $y$ works.

(b) Prove that there exists $x \in \mathbb{R}$ such that for every $y \leq -3$ we have $xy \geq 6$.

Proof. We figure out a specific value of $x$ which works. Motivated by the fact that $(-2) \cdot (-3) = 6$, we try the following. Let $x = -2$. Then for every $y \leq -3$ we have

$$xy = (-2) y \geq (-2) (-3) = 6$$

because multiplying the inequality $y \leq -3$ by $-2$ reverses it. This proves the statement.