Finding the maximum and minimum of a continuous function $f$ on a closed bounded domain $\mathcal{D}$

Consider the function
\[ f(x, y) = x^3 + y^3 - 3xy \]
on the closed square $\mathcal{D}$ given by $-1 \leq x \leq 2$, $-1 \leq y \leq 2$. Where do the maximum and minimum occur?

(1) Find the interior critical points.
(2) For each side of the boundary, find the extrema for that side.
(3) Among all the interior critical points and extrema on the boundary, find the maximum and minimum.

(1) The partial derivatives of $f$ are:
\[ f_x = 3x^2 - 3y, \]
\[ f_y = 3y^2 - 3x. \]

A point $(x, y)$ is critical for $f$ exactly when
\[ x^2 = y, \]
\[ y^2 = x. \]

We solve this:
\[ x = y^2 = x^4, \quad \text{i.e., } x (x^3 - 1) = 0. \]

So $x = 0$ (implies $y = 0$) or $x = 1$ (implies $y = 1$). So the critical points of $f$ are $(0, 0)$ and $(1, 1)$. Both of these points are in the domain $\mathcal{D}$.

The values of $f$ at these points are
\[ f(0, 0) = 0, \]
\[ f(1, 1) = -1. \]

(2)

(Lo) Consider the LOWER side of the boundary. That is, $y = -1$ and $-1 \leq x \leq 2$. There we have
\[ f(x, -1) = x^3 - 1 + 3x, \quad -1 \leq x \leq 2. \]
We compute
\[
\frac{d}{dx} (f(x, -1)) = 3x^2 + 3.
\]
So \(\frac{d}{dx} (f(x, -1)) \neq 0\) for \(-1 \leq x \leq 2\). This means that the extrema of \(f(x, -1), -1 \leq x \leq 2\), occur at the endpoints \(x = -1\) and \(x = 2\). We have
\[
f(-1, -1) = -5,
\]
\[
f(2, -1) = 13.
\]

(Up) Consider the UPPER side of the boundary. That is, \(y = 2\) and \(-1 \leq x \leq 2\). There we have
\[
f(x, 2) = x^3 + 8 - 6x, \quad -1 \leq x \leq 2.
\]
We compute
\[
\frac{d}{dx} (f(x, -1)) = 3x^2 - 6.
\]
So \(\frac{d}{dx} (f(x, -1)) = 0\) when \(x = \pm \sqrt{2}\). Only \(\sqrt{2}\) is between \(-1\) and 2. This means that the extrema of \(f(x, 2), -1 \leq x \leq 2\), could occur at \(x = -1, x = \sqrt{2}\), and/or \(x = 2\). At these points we have
\[
f(-1, 2) = 13,
\]
\[
f(\sqrt{2}, 2) = 8 - 4\sqrt{2} \approx 2.343,
\]
\[
f(2, 2) = 4.
\]
Since we have the symmetry \(f(y, x) = f(x, y)\), the analogous statements are true for the LEFT and RIGHT sides of the boundary because switching \(x\) and \(y\) takes the LOWER side to the LEFT side and takes the UPPER side to the RIGHT side. So we get

(Le) On the LEFT side the extrema of \(f(-1, y), -1 \leq y \leq 2\), could only occur at the endpoints \(y = -1\) and \(y = 2\), at which
\[
f(-1, -1) = -5,
\]
\[
f(-1, 2) = 13.
\]

(Ri) On the RIGHT side the extrema of \(f(2, y), -1 \leq y \leq 2\), could only occur at \(y = -1, y = \sqrt{2}\), and/or \(y = 2\), at which
\[
f(2, -1) = 13,
\]
\[
f(2, \sqrt{2}) = 8 - 4\sqrt{2} \approx 2.343,
\]
\[
f(2, 2) = 4.
\]

(3) Among the above values \(-5, -1, 0, 2.343, 4, 13\) obtained from parts (1) and (2), the largest is 13 and the smallest is \(-5\). So the maximum of 13 occurs at \((2, -1)\) and \((-1, 2)\). The minimum of \(-5\) occurs at \((-1, -1)\).