Section 15.2

Let $\mathcal{D}$ be the region between the graph of two functions $g$ and $h$ on the interval $[a,b]$, that is, $\mathcal{D}$ is the set of $(x,y)$ such that $g(x) \leq y \leq h(x)$ and $a \leq x \leq b$.

The integral of a function $f(x,y)$ over $\mathcal{D}$ can be computed as

$$\iint_{\mathcal{D}} f(x,y) \, dA = \int_{a}^{b} \int_{g(x)}^{h(x)} f(x,y) \, dy \, dx.$$  

Let $\mathcal{S}$ be the solid region under graph of $f(x,y)$ over the domain $\mathcal{D}$. Slicing $\mathcal{S}$ at fixed $x$ yields the plane region over the interval $g(x) \leq y \leq h(x)$ under the graph of $f(x,y)$. The area $A(x)$ of this plane region is

$$A(x) = \int_{g(x)}^{h(x)} f(x,y) \, dy.$$  

So the volume of $\mathcal{S}$ is

$$\int_{a}^{b} A(x) \, dx = \int_{a}^{b} \int_{g(x)}^{h(x)} f(x,y) \, dy \, dx.$$  

**Example 1.** Compute the double integral of $f(x,y) = x$ over the region $\mathcal{D}$ given by $0 \leq x \leq \pi$ and $-x \leq y \leq \sin(x^2)$. 


Answer:

\[
\int_{0}^{\pi} \int_{-x}^{\sin(x^2)} x \, dy \, dx = \int_{0}^{\pi} (x \sin(x^2) + x^2) \, dx = \frac{1}{3} \pi^3 - \frac{1}{2} \cos(\pi^2) + \frac{1}{2} \approx 11.287.
\]

**Example 2.** Consider the integral

\[
\int_{0}^{1} \int_{y}^{1} \sin x \, dy \, dx.
\]

The problem is that one cannot write \( \int \sin x \, dx \) in terms of elementary functions. So we try switching the order of integration. Now the domain \( D \) is \( y \leq x \leq 1 \) and \( 0 \leq y \leq 1 \).

We can describe \( D \) in the reverse order: \( 0 \leq y \leq x \) and \( 0 \leq x \leq 1 \). So

\[
\int_{0}^{1} \int_{y}^{1} \frac{\sin x}{x} \, dy \, dx = \int_{0}^{1} \int_{0}^{x} \frac{\sin x}{y} \, dy \, dx.
\]

Now \( \int_{0}^{x} \frac{\sin x}{y} \, dy = \sin x \). So

\[
\int_{0}^{1} \int_{0}^{x} \frac{\sin x}{y} \, dy \, dx = \int_{0}^{1} \sin x \, dx = 1 - \cos 1 \approx 0.460.
\]
Example 3. Consider
\[ \int_0^1 \int_{x^{2/3}}^1 xe^{y^4} dy dx. \]
The problem is that we do not know how evaluate
\[ \int_{x^{2/3}}^1 xe^{y^4} dy = x \int_{x^{2/3}}^1 e^{y^4} dy. \]
So again we switch the order of integration. The domain \( D \) is given by \( x^{2/3} \leq y \leq 1 \) and \( 0 \leq x \leq 1 \).

We rewrite this as \( D \) is given by \( 0 \leq x \leq y^{3/2} \) and \( 0 \leq y \leq 1 \). So
\[ \int_0^1 \int_{x^{2/3}}^1 xe^{y^4} dy dx = \int_0^1 \int_0^{y^{3/2}} xe^{y^4} dxdy. \]
Now
\[ \int_0^{y^{3/2}} xe^{y^4} dx = e^{y^4} \int_0^{y^{3/2}} xdx = \frac{1}{2} y^3 e^{y^4}. \]
Hence
\[ \int_0^1 \int_0^{y^{3/2}} xe^{y^4} dxdy = \int_0^1 \frac{1}{2} y^3 e^{y^4} dy = \frac{1}{8} (e - 1). \]