Let \( z = f(x, y) \). The gradient of \( f \) at \( P = (x, y) \) is
\[
\nabla f_P = (f_x(P), f_y(P)).
\]
We have the properties:

(i) \( \nabla (f + g) = \nabla f + \nabla g \),
(ii) \( \nabla (cf) = c \nabla f \),
(ii) \( \nabla (fg) = g \nabla f + f \nabla g \).

Chain Rule 1: If \( F(t) \) is a function of one variable, then
\[
\nabla (F(f(P))) = F'(f(P)) \nabla f_P.
\]
Chain Rule 2: If \( \vec{c}(t) = \langle x(t), y(t) \rangle \), then
\[
\frac{d}{dt} f(\vec{c}(t)) = \frac{d}{dt} f(x(t), y(t)) = \nabla f_{\vec{c}(t)} \cdot \vec{c}'(t).
\]
Directional derivative: If \( \vec{u} = \langle h, k \rangle \) is a unit vector, then the directional derivative of \( f \) in the direction \( \vec{u} \) is
\[
D_{\vec{u}}f = \nabla f \cdot \vec{u}.
\]
If \( \vec{v} \) is not unit, then the directional derivative of \( f \) in the direction \( \vec{v} \) is:
\[
D_{\vec{u}}f = \nabla f \cdot \vec{u}, \quad \text{where} \quad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|}.
\]
The directional derivative can also be defined as: Let \( \vec{c}(t) = P + t\vec{u} \), so that \( \vec{c}'(t) = \vec{u} \). Then
\[
\frac{d}{dt} f(\vec{c}(t)) = \nabla f_{\vec{c}(t)} \cdot \vec{c}'(t) = \nabla f \cdot \vec{u}.
\]
Useful formula:
\[
D_{\vec{u}}f = \nabla f \cdot \vec{u} = \|\nabla f\| \cos \theta, \quad \text{where} \quad \theta \text{ is the angle between } \nabla f \text{ and } \vec{u}.
\]
Consequence: The maximum directional derivative is when \( \theta = 0 \). Here (assume \( \|\nabla f\| \neq 0 \)), \( \vec{u} = \frac{\nabla f}{\|\nabla f\|} \) and \( D_{\vec{u}}f = \|\nabla f\| \).

The minimum directional derivative is when \( \theta = \pi \). Here (assume \( \|\nabla f\| \neq 0 \)), \( \vec{u} = -\frac{\nabla f}{\|\nabla f\|} \) and \( D_{\vec{u}}f = -\|\nabla f\| \).