Math 140B  HW5, due Friday February 24 at 2pm

#1. We say that a family $\mathcal{G}$ of functions from a set $E$ to a metric space $(Y,d)$ is **pointwise bounded** if there exists $y_0 \in Y$ and a function $\phi : E \to \mathbb{R}$ such that $d(f(x), y_0) \leq \phi(x)$ for all $x \in E$ and all $f \in \mathcal{G}$.

We say that $\mathcal{G}$ is **uniformly bounded** if there exists $y_0 \in Y$ and $M \in \mathbb{R}$ such that $d(f(x), y_0) \leq M$ for all $x \in E$ and all $f \in \mathcal{G}$.

Let $\mathcal{F}$ be a pointwise bounded family of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that for each $k \in \mathbb{Z}^+$ the family $\mathcal{F}_k = \{f|[-k,k] \ | f \in \mathcal{F}\}$ is an equicontinuous family of functions on $[-k,k]$. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions with $f_n \in \mathcal{F}$ for each $n \in \mathbb{Z}^+$. Prove that there exists a subsequence $\{f_{n_k}\}$ with the property that there exists a continuous function $f$ such that $f_{n_k}$ converges to $f$ pointwise on $\mathbb{R}$ and, for any $M \in \mathbb{R}^+$, $f_{n_k}$ converges uniformly to $f$ on $[-M,M]$.

#2. Let $m \in \mathbb{Z}^+$ and let $\mathcal{F}$ be a pointwise bounded family of differentiable functions from $\mathbb{R}^m$ to $\mathbb{R}$. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function. Suppose that for every $f \in \mathcal{F}$ we have $|\frac{\partial f}{\partial x^i}(x)| \leq \phi(|x|)$ for all $x \in \mathbb{R}^m$. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions with $f_n \in \mathcal{F}$ for each $n \in \mathbb{Z}^+$. Prove that there exists a subsequence $\{f_{n_k}\}$ with the property that there exists a continuous function $f$ such that $f_{n_k}$ converges to $f$ pointwise on $\mathbb{R}^m$ and, for any $M \in \mathbb{R}^+$, $f_{n_k}$ converges uniformly to $f$ on $B_M(0)$, where $0$ is the origin in $\mathbb{R}^m$.

#3. Let $\mathcal{F}$ be a family of twice differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that there exists a sequence of real numbers $r_m \rightarrow +\infty$ and a sequence of positive numbers $b_m$ such that

$$|f(x)| \leq b_m, \ |f'(x)| \leq b_m, \text{ and } |f''(x)| \leq b_m$$

for all $f \in \mathcal{F}$, $-r_m \leq x \leq r_m$, and $m \in \mathbb{Z}^+$. Prove that for any sequence of functions $\{f_n\}_{n=1}^\infty$ in $\mathcal{F}$ there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ with the property that there exists a continuous function $f$ such that

- (a) $f_{n_k}$ converges to $f$ pointwise on $\mathbb{R}$,
- (b) $f'_{n_k}$ converges to $f'$ pointwise on $\mathbb{R}$,
- (c) for any $M \in \mathbb{R}^+$, $f_{n_k}$ converges uniformly to $f$ on $[-M,M]$,
- (d) for any $M \in \mathbb{R}^+$, $f''_{n_k}$ converges uniformly to $f''$ on $[-M,M]$.

#4. In Problem #3 suppose we remove the condition that $|f'(x)| \leq b_m$ from the hypothesis. Does the conclusion remain true?

#5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $S \subset \mathbb{R}$ be a subset such that $0 \not\in \bar{S}$ and $0 \not\in S$. Given $t \in \mathbb{R} - \{0\}$, define $f_t : \mathbb{R}^2 \to \mathbb{R}$ by $f_t(x,y) = f(t^{-1}x, t^{-1}y)$. Define the family of functions $\mathcal{F} = \{f_t\}_{t \in S}$. Suppose that $\mathcal{F}$ is an equicontinuous family of functions on $\mathbb{R}^2$. What can you say about the function $f$? I.e., what property must $f$ have? Prove it.

#6. In Problem #5 suppose that $f_t$ is defined instead by $f_t(x,y) = f(t^{-1}x, y)$. What is the answer now?