HW #1 SOLUTIONS

1. Let $f : [a, b] \to \mathbb{R}$ and let $v : I \to \mathbb{R}$, where $I$ is an interval containing $f([a, b])$. Suppose that $f$ is continuous at $x \in [a, b]$. Suppose that $\lim_{x \to f(x)} v(s) = v(f(x))$. Using the $\epsilon - \delta$ definition of limit, prove that $\lim_{t \to x} v(f(t)) = v(f(x))$.

Solution.

Take an $\epsilon > 0$. Since $\lim_{x \to f(x)} v(s) = v(f(x))$, there exists $\epsilon_1 > 0$ such that if $s \in I$ and $|s - f(x)| < \epsilon_1$ then $|v(s) - v(f(x))| < \epsilon$. Now, $f$ is continuous at $x$, so there exists $\delta > 0$ such that if $t \in [a, b]$ is such that $|t - x| < \delta$ then $|f(t) - f(x)| < \epsilon_1$. Since $I$ contains $f([a, b])$, we automatically have $f(t) \in I$, and also $|f(t) - f(x)| < \epsilon_1$ from above, so $|v(f(t)) - v(f(x))| < \epsilon$ as desired. This shows that $\lim_{t \to x} v(f(t)) = v(f(x))$, i.e. that the composition $v \circ f$ is continuous at $x$.

2. Let $g : I \to \mathbb{R}$ be a function, where $I$ is an interval containing 0 in its interior. Suppose $\lim_{x \to 0} \frac{g(x)}{x} = 0$. Define

$$f(x) = \begin{cases} g(x) \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that $f'(0) = 0$.

Solution.

By definition, $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{g(x) \sin \frac{1}{x}}{x}$. We have the inequalities $0 \leq \left| \frac{g(x) \sin \frac{1}{x}}{x} \right| \leq \left| \frac{g(x)}{x} \right|$ for $x \neq 0$, since $|\sin y| \leq 1$ for all $y \in \mathbb{R}$, in particular for $y = \frac{1}{x}$. Since we are given that $\lim_{x \to 0} \frac{g(x)}{x} = 0$, we also have $\lim_{x \to 0} \left| \frac{g(x)}{x} \right| = 0$. Of course, $\lim_{x \to 0} 0 = 0$, so by the Squeeze Theorem, $\lim_{x \to 0} \frac{g(x) \sin \frac{1}{x}}{x} = 0$, or $\lim_{x \to 0} \frac{g(x) \sin \frac{1}{x}}{x} = 0$.

Remark. We’re using the easy-to-check fact that for a function $h : (a, b) \to \mathbb{R}$ and $t \in (a, b)$, $\lim_{x \to t} h(x) = 0$ if and only if $\lim_{x \to t} |h(x)| = 0$. To see this, we need only unravel the definition of limit in this context: $\lim_{x \to t} h(x) = 0$ says that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - t| < \delta$ then $|h(x) - 0| < \epsilon$, whereas $\lim_{x \to t} |h(x)| = 0$ says that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - t| < \delta$ then $||h(x)| - 0| < \epsilon$. Note however that $||h(x)| - 0| = |h(x)| = |h(x) - 0|$ so the two statements are saying the same thing.

3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions satisfying $f'(x) > g'(x)$ for all $x \in \mathbb{R}$ and $f(0) = g(0)$. Prove that $f(x) > g(x)$ for $x > 0$ and $f(x) < g(x)$ for $x < 0$.

Solution.

Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = f(x) - g(x)$ for $x \in \mathbb{R}$. Then $h(0) = f(0) - g(0) = 0$ and $h'(x) = f'(x) - g'(x)$.
\( f'(x) - g'(x) > 0 \) if \( x \in \mathbb{R} \). If \( x > 0 \), then by the Mean Value Theorem, \( h(x) = h(x) - h(0) = h'(c)(x - 0) \) for some \( c \) between 0 and \( x \). But then \( h'(c) = f'(c) - g'(c) > 0 \), so \( h(x) = h'(c)x > 0 \).

If \( x < 0 \), then the same computation \( h(x) = h'(c)x \) for some \( c \) between 0 and \( x \) shows that \( h(x) < 0 \) because \( h'(c) > 0 \) and \( x < 0 \).

4. Rudin Ch 5. Exercise #7. Suppose \( f'(x), g'(x) \) exist, \( g'(x) \neq 0 \), and \( f(x) = g(x) = 0 \). Prove that

\[
\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.
\]

This holds also for complex functions.

**Solution.**

\[
\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f(t) - f(x)}{g(t) - g(x)} \cdot \frac{g(t) - g(x)}{t - x},
\]

and we are given that \( \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x) \) and \( \lim_{t \to x} \frac{g(t) - g(x)}{t - x} = g'(x) \neq 0 \)

both exist, hence the ratio \( \lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)} \) (note that we need \( g'(x) = 0 \)).

5. Rudin Ch 5. Exercise #13. Suppose \( a \) and \( c \) are real numbers, \( c > 0 \), and \( f \) is defined on \([-1, 1]\) by

\[
f(x) = \begin{cases} 
|x|^a \sin(|x|^{-c}) & \text{(if } x \neq 0\text{),} \\
0 & \text{(if } x = 0\text{).}
\end{cases}
\]

Prove the following statements:

(a) \( f \) is continuous if and only if \( a > 0 \).
(b) \( f'(0) \) exists if and only if \( a > 1 \).
(c) \( f' \) is bounded if and only if \( a \geq 1 + c \).
(d) \( f' \) is continuous if and only if \( a > 1 + c \).
(e) \( f''(0) \) exists if and only if \( a > 2 + c \).
(f) \( f'' \) is bounded if and only if \( a > 2 + 2c \).
(g) \( f'' \) is continuous if and only if \( a > 2 + 2c \).

**Solution.**

(a) Suppose \( a \leq 0 \). Then for each positive integer \( k \), consider a real number \( x_k = \left( \frac{1}{2 + 2\pi k} \right)^{1/c} \in (0, 1) \subset [-1, 1] \).

Then, \( f(x_k) = |x_k|^a \sin(|x_k|^{-c}) = \left( \frac{1}{2 + 2\pi k} \right)^{a/c} \sin\left( \frac{\pi}{2} + 2\pi k \right) = \left( \frac{\pi}{2} + 2\pi k \right)^{-a/c} \). If \( a = 0 \), then \( \lim_{k \to \infty} f(x_k) = 1 \), whereas if \( a < 0 \) then \( \lim_{k \to \infty} f(x_k) = \infty \). In either case, we have \( \lim_{k \to \infty} f(x_k) \neq f(0) \), whereas \( \lim_{k \to \infty} x_k = 0 \) (\( c > 0 \)). Therefore \( f \) is not continuous at 0 (recall that a function \( f \) defined on a neighborhood of 0 is continuous at 0 if and only if \( \lim_{x \to 0} f(x) = f(0) \) for all sequences \( \{x_k\}_k \) such that \( \lim_{k \to \infty} x_k = 0 \)). In fact, we have shown that \( f \) has neither the left nor the right limit at 0.

Suppose now that \( a > 0 \). Then if \( x \neq 0 \), we have \( |f(x)| = |x|^a \sin(|x|^{-c}) \leq |x|^a \), and for \( x = 0 \) of course we have the same inequality (since \( a > 0 \), \( 0^a = 0 \) is defined also of course). Since \( \lim_{x \to 0} |x|^a = 0 \), we have \( \lim_{x \to 0} |f(x)| = 0 \) by the Squeeze Theorem.
(b) \( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \frac{|x|^{a-1} \sin(|x|^{-c})}{x} \) if the limit exists.

If \( a \leq 1 \), then \( \lim_{x \to 0^+} x^{a-1} \sin(|x|^{-c}) \) does not exist, as in part (a) by considering the sequence \( x_k = \left( \frac{1}{2 + \pi k} \right)^{1/c} \).

Therefore we need \( a > 1 \). In this case, \( \left| \frac{|x|^{a-1} \sin(|x|^{-c})}{x} \right| = |x|^{a-1} \sin(|x|^{-c}) \leq |x|^{a-1} \) and the latter tends to 0 as \( x \to 0 \), so \( f'(0) \) exists by the Squeeze Theorem and \( f'(0) = 0 \) in fact in this case.

(c) Note first that \( f(x) = f(-x) \) if \( x \in [-1, 1] \), i.e. \( f \) is an even function. For \( x \neq 0 \), \( f \) is certainly differentiable at \( x \) since \( f \) is given by the expression \( x^a \sin(x^{-c}) \) if \( x > 0 \) on a neighborhood of \( x \) not containing 0 and by \( (-x)^a \sin((-x)^{-c}) \) on a neighborhood of \( x \) not containing 0 and both expressions are products of compositions of differentiable functions, hence differentiable. We assume that \( a > 1 \) from part (b) to discuss the existence of \( f' \) on \([-1, 1]\) in the first place.

For \( x > 0 \), \( f'(x) = ax^{a-1} \sin(x^{-c}) + x^a \cos(x^{-c})(-cx^{-c-1}) = ax^{a-1} \sin(x^{-c}) - cx^{a-1} \cos(x^{-c}) \) (recall the remark above that \( f \) is given by \( x \mapsto x^a \sin(x^{-c}) \) in a neighborhood of \( x \) in \([-1, 1]\) not containing 0). Also for \( a > 1 \), \( f'(0) \) exists and \( f'(0) = 0 \).

For \( x < 0 \), \( f \) is again differentiable at \( x \) because \( f(x) = f(-x) \) expresses \( f \) as a composition of the differentiable function \( f|_{(0,1]} \) (this is the restriction of \( f \) on \((0,1]\)) together with the differentiable function \( x \mapsto -x \) on a neighborhood of \( x \) not containing 0. Using the chain rule now, we have \( f'(x) = -f'(-x) \), so that

\[
f'(x) = \begin{cases} 
ax^{a-1} \sin(x^{-c}) - cx^{a-1} \cos(x^{-c}) & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
c|x|^{a-c-1} \cos(|x|^{-c}) - a|x|^{a-1} \sin(|x|^{-c}) & \text{if } x < 0
\end{cases}
\]

At any rate, we then need only analyze the boundedness of the expression \( ax^{a-1} \sin(x^{-c}) - cx^{a-1} \cos(x^{-c}) \) on \((0,1]\) (for \( 0 < x \leq 1 \)). Since \( a > 1 \), the first term has absolute value \( |ax^{a-1} \sin(x^{-c})| \leq a|x|^{a-1} \), which tends to 0 as \( x \to 0^+ \) because \( a > 1 \). It is thus bounded. For the latter term, \( |cx^{a-1} \cos(x^{-c})| \leq cx^{a-1} \), and this tends to 0 as \( x \to 0^+ \) if \( a \geq 1 + c \). In the case \( a < 1 + c \), we take the sequence \( \{x_k\}_{k \geq 1} \) defined by \( x_k = (\frac{1}{2\pi})^{1/c} \in (0,1] \), which tends to 0 as \( k \to \infty \), and yet \( f(x_k) = -c(\frac{1}{2\pi})^{(a-c-1)/c} \to -\infty \) as \( k \to \infty \), showing that \( f \) is not bounded on \((0,1]\) if \( a < 1 + c \).

(d) We assume \( f' \) exists throughout \([-1,1] \), i.e. \( a > 1 \).

Assume first that \( a > 1 + c \), then from \( f'(x) = -f'(-x) \) again, we need only show that \( \lim_{x \to 0^+} f'(x) = f'(0) = 0 \) to check continuity of \( f' \) (it is continuous on \((0,1]\) because it is expressed on a neighborhood of a point in \((0,1]\) by a combination of products, addition, and composition of differentiable functions). \( \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} (ax^{a-1} \sin(x^{-c}) - cx^{a-1} \cos(x^{-c})) \), and the first term has limit \( \lim_{x \to 0^+} ax^{a-1} \sin(x^{-c}) = 0 \) since \( a > 1 \) (Squeeze Theorem yet again here). For the latter term, note that \( \lim_{x \to 0^+} -cx^{a-1} \cos(x^{-c}) = 0 \) again by the Squeeze Theorem because \( a > 1 + c \).

Assume now that \( f' \) is continuous on \([-1,1] \). Then \( f'(|-1,1]| \) is compact hence bounded, so part (c) implies that \( a \geq 1 + c \). We need only rule out the case \( a = 1 + c \) then to show that \( a > 1 + c \). So suppose \( a = 1 + c \). Then for \( x > 0 \), \( f'(x) = ax \sin(x^{-c}) - c \cos(x^{-c}) \). Now take the sequence \( x_k = (2\pi k)^{-1/c} \), which has the property that \( x_k \to 0^+ \) as \( k \to \infty \), and \( f'(x_k) = -c \) for all \( k \). That is, \( \lim_{k \to \infty} f'(x_k) = -c \neq 0 = f'(0) \), so \( f' \) is not continuous at 0.

(e) We assume here that \( f' \) is continuous throughout \([-1,1] \), i.e. \( a > 1 + c \) by part (d).

\[
f''(0) = \lim_{x \to 0} \frac{f''(x) - f''(0)}{x - 0}.
\]

For \( x < 0 \), \( \frac{f'(x) - f'(0)}{x - 0} = \frac{f'(x) - f'(0)}{x - 0} = \frac{f'(x)}{x} \), and for \( x > 0 \), \( \frac{f'(x) - f'(0)}{x - 0} = \frac{f'(x)}{x} \).

\[
f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0}.
\]
Suppose \( a < 2 + c \). Then take the sequence \( x_k = (2\pi k)^{-1/c} \). Then \( x_k \to 0 \) as \( k \to \infty \) whereas \( g(x_k) = -c(2\pi k)(c+2-a)/c \), so that \( \lim_{k \to \infty} g(x_k) = -\infty \), so \( f''(0) \) does not exist.

Suppose \( a = 2 + c \). Then take \( x_k = (\pi k)^{-1/c} \), so \( g(x_k) = -c(-1)^k \), and \( \lim_{k \to \infty} g(x_k) \) does not exist.

Suppose \( a > 2 + c \). Then the first term \( |ax^{a-2} \sin(x^{-c})| \leq ax^{a-2} \) and this tends to 0 as \( x \to 0 \). For the second term, \( |cx^{a-c-2} \sin(x^{-c})| \leq cx^{a-c-2} \) and again this tends to 0 as \( x \to 0 \). So by the Squeeze Theorem, \( f''(0) \) exists.

(f) Assume now that \( a > 2 + c \). We know from the above computation in part (e) that \( f''(0) = 0 \). \( f' \) is expressed by differentiate function on \([-1, 0) \cup (0, 1]\), so \( f' \) is differentiable on it. Therefore with the assumption that \( a > 2 + c \), \( f' \) is differentiable on \([-1, 1]\). We have \( f'(x) = -f'(-x) \), so by the Chain Rule, \( f''(x) = f''(-x) \), i.e. \( f'' \) is an even function and it suffices to compute \( f''(x) \) for \( x > 0 \), i.e. 

\[
\frac{d}{dx}(ax^{a-1} \sin(x^{-c}) - cx^{a-c-1} \cos(x^{-c})) = (a(a-1)x^{a-2} - c^2x^{a-2c-2}) \sin(x^{-c}) - (-2a + c + 1)x^{a-c-2} \cos(x^{-c}).
\]

Note that both \( a(a-1) > 0 \) and \((-2a + c + 1) > 0\) from \( a > 2 + c \) and \( c > 0 \). All the terms appearing here except for \( ax^{a-2c-2} \sin(x^{-c}) \) tends to 0 as \( x \to 0 \), so \( f'' \) is bounded on \([-1, 1]\) if and only if \( x^{a-2c-2} \sin(x^{-c}) \) is bounded on \((0, 1]\). By the same argument as in the previous parts, this happens exactly when \( a - 2c - 2 \geq 0 \), or when \( a > 2 + 2c \).

(g) Suppose \( f'' \) is continuous on \([-1, 1]\). Then \( f''([-1, 1]) \) is compact since \([-1, 1]\) is compact (closed and bounded, Heine-Borel). So by part (f), \( f'' \) is bounded on \([-1, 1]\) and thus \( a \geq 2 + 2c \). We rule out the case \( a = 2 + 2c \). If \( a = 2 + 2c \), we have for \( x > 0 \), \( f''(x) = (a(a-1)x^{a-2} - c^2x^{a-2c-2}) \sin(x^{-c}) - (-2a + c + 1)x^{a-2c-2} \cos(x^{-c}). \)

Set \( x_k = (\frac{a}{2} + 2\pi k)^{-1/c} \) as usual, then \( f''(x_k) = a(a-1)(\frac{a}{2} + 2\pi k)^{-(a-2)/c} - c^2 \), which does not tend to 0 as \( k \to \infty \). Since \( f''(0) = 0 \) and \( \lim_{k \to \infty} x_k = 0 \), \( f'' \) is not continuous at 0.

Suppose \( a > 2 + 2c \). Then all the terms in the expression above for \( f''(x) \) for \( x > 0 \) tends to 0 as \( x \to 0 \) by the Squeeze Theorem, so \( f'' \) is continuous at 0. It’s continuous everywhere else so \( f'' \) is continuous on \([-1, 1]\).

6. Rudin Ch 5. Exercise #14. Let \( f \) be a differentiable real function defined in \((a, b)\). Prove that \( f \) is convex if and only if \( f' \) is monotonically increasing. Assume next that \( f''(x) \) exists for every \( x \in (a, b) \), and prove that \( f \) is convex if and only if \( f''(x) \geq 0 \) for all \( x \in (a, b) \).

**Solution.**

Recall that a real function \( f \) defined on \((a, b)\) is convex if and only if for all \( x < y < z \) in \((a, b)\), 

\[
f(y) - f(x) \leq \frac{f(z) - f(y)}{z-y} \cdot (y-x).
\]

We will prove this equivalence at the end of this solution.

Suppose that \( f \) is convex. Take \( x < y \) in \((a, b)\). Take sequences \( \{x_n\}_n \) and \( \{y_n\}_n \) such that \( x < x_n < y_n < y \) for all \( n \), \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \). Since \( f \) is assumed to be differentiable, we then have \( f'(x) = \lim_{n \to \infty} a_n \) and \( f'(y) = \lim_{n \to \infty} b_n \) if \( a_n = \frac{f(x_n) - f(x)}{x_n - x} \) and \( b_n = \frac{f(y_n) - f(y)}{y_n - y} \). By the convexity, \( a_n = \frac{f(x_n) - f(x)}{x_n - x} \leq \frac{f(y_n) - f(x)}{y_n - x} \leq b_n \) for all \( n \geq 1 \). Therefore \( f'(x) = \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n = f'(y) \). That is, \( f' \) is monotonically increasing.

Suppose now that \( f' \) is monotonically increasing. Take \( x < y < z \) in \((a, b)\). By the Mean Value Theorem,
\[ \frac{f(y) - f(x)}{y - x} = f'(c) \text{ for some } c \in (x, y) \text{ and } \frac{f(z) - f(y)}{z - y} = f'(d) \text{ for some } d \in (y, z). \] Then \( x < c < y < d < z \), and in particular \( c < d \). Therefore \( f'(c) \leq f'(d) \), and so \( f \) is convex.

Assuming that \( f''(x) \) exists for all \( x \in (a, b) \), note only that \( f' \) is monotonically increasing if and only if \( f''(x) \geq 0 \) for all \( x \in (a, b) \) by the Mean Value Theorem. One direction follows from Theorem 5.11. Suppose that \( f' \) is monotonically increasing. Then fix \( y \in (a, b) \) and let \( h : [y, b) \to \mathbb{R} \) be defined by \( h(x) = f'(x) - f'(y) \). Then \( h(y) = 0, h \) is differentiable since \( f' \) is differentiable, and \( h(x) = h(x) - h(y) = h'(c)(x - y) \) for some \( c \in (y, x) \). But \( h'(c) = f''(c) \geq 0 \) (we’re fixing \( y \)), so that \( h(x) \geq 0 \). That is, \( f'(x) - f'(y) \geq 0 \), or \( f'(x) \geq f'(y) \). So \( f' \) is monotonically increasing.

**Proof of equivalent definitions of convexity.** We compare the two conditions on a function \( f : (a, b) \to \mathbb{R} \): i) For all \( t \in (0, 1) \) and \( x, y \in (a, b) \), \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \) and ii) For all \( x, y, z \in (a, b) \) such that \( x \leq y < z \), \( \frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y} \).

i) implies ii). Take \( x < y < z \) in \( (a, b) \). We take here \( t = \frac{z - y}{x - y} \), so that \( t \in (0, 1) \). Then \( y = tx + (1-t)y \), and so \( f(y) \leq tf(x)+(1-t)f(z) = \frac{z - y}{x - y}f(x) + \frac{y - z}{x - y}f(z) \). Rearranging this inequality gives precisely \( \frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y} \).

ii) implies i). If \( t \in (0, 1) \) and \( x < y \) in \( (a, b) \), then \( x < tx + (1-t)y < y \). So using the condition ii), we have \( \frac{f(tx + (1-t)y) - f(x)}{tx + (1-t)y - x} \leq \frac{f(y) - f(tx + (1-t)y)}{y - (tx + (1-t)y)} \). Rearranging this inequality gives precisely that \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \).

7. Rudin Ch 5, Exercise #15. Suppose \( a \in \mathbb{R}^1, f \) is twice-differentiable real function on \((a, \infty)\), and \( M_0, M_1, M_2 \) are the least upper bounds of \( |f(x)|, |f'(x)|, |f''(x)| \), respectively, on \((a, \infty)\). Prove that \( M_1^2 \leq 4M_0M_2 \). Does this hold for vector valued functions too?

**Solution.**

If \( x \in (a, \infty) \) and \( h > 0 \), then by the theorem of Taylor, there is \( \zeta \in (x, x + h) \) such that \( f(x + h) = f(x) + f'(x)h + \frac{f''(\zeta)}{2}h^2 \). Or \( f'(x)h = f(x + h) - f(x) - \frac{f''(\zeta)}{2}h^2 \). Therefore \( |f'(x)h| \leq M_0 + M_0 + \frac{M_2}{2}h^2 \).

We have so far that \( \frac{M_2}{2}h^2 = M_1h + 2M_0 \geq 0 \) for all \( h \in \mathbb{R} \). Considering the real valued function \( g(h) = \frac{M_2}{2}h^2 - M_1h + 2M_0 \) defined on \( \mathbb{R} \), a degree 2 polynomial, the fact that it is always nonnegative implies that it has at most 1 root, i.e. that its discriminant \( M_1^2 - 4\left(\frac{M_2}{2}\right)(2M_0) \leq 0 \), or \( M_1^2 \leq 4M_0M_2 \).

Let’s consider now a vector valued, twice differentiable function \( f : (a, \infty) \to \mathbb{R}^n \). We reduce the problem to the one-variable case by considering various projections of the curve \( f \) onto the 1-dimensional subspaces of \( \mathbb{R}^n \). To that end, take a direction \( u \in S^{n-1} = \{ (a_1, a_2, \ldots, a_n) \mid \sum_{i=1}^n a_i^2 = 1 \} \). We then consider the projection of the curve \( f \) onto the line spanned by \( u \), i.e. consider the function \( g : (a, \infty) \to \mathbb{R} \) by \( g(t) = u \cdot f(t) = |u||f(t)| \cos \theta \) where \( \theta \) is the “angle” between \( u \) and \( f(t) \).

Using the one-variable case, we have \( \sup_{t \in (a, \infty)} |g'(t)|^2 \leq 4 \sup_{t \in (a, \infty)} |g(t)| \sup_{t \in (a, \infty)} |g''(t)| \). Note that \( g'(t) = u \cdot f'(t) \) and \( g''(t) = u \cdot f''(t) \), so by the Cauchy-Schwartz inequality we have \( |g'(t)| \leq |u||f'(t)| = |f'(t)| \) and likewise \( |g''(t)| \leq |f''(t)| \).
We have thus shown that for all \( u \in \mathbb{S}^{n-1} \), \((\sup_{t \in (a, \infty)} |u \cdot f'(t)|)^2 \leq 4M_0M_2\). Note that \((\sup_{t \in (a, \infty)} |f'(t)|)^2 = \sup_{t \in (a, \infty)} |f'(t)|^2\). Now take any \( \epsilon > 0 \). Then there exists \( \zeta \in (a, \infty) \) such that \( \sup_{t \in (a, \infty)} |f'(t)| - \epsilon < |f'(\zeta)|\). We now then choose \( u \in \mathbb{S}^{n-1} \) to maximize the magnitude of the projection \( f'(\zeta) \) onto \( u \), i.e. take \( u = \frac{f'(\zeta)}{|f'(\zeta)|} \) in the direction parallel to \( f'(\zeta) \). Then we have the inequalities
\[
(\sup_{t \in (a, \infty)} |f'(t)| - \epsilon)^2 \leq |f'(\zeta)|^2 = |u \cdot f'(\zeta)|^2 \leq (\sup_{t \in (a, \infty)} |u \cdot f'(t)|)^2 \leq 4M_0M_2
\]
Since \( \epsilon > 0 \) was arbitrary, we have \( M_1^2 \leq 4M_0M_2\).

Now take
\[
f(x) = \begin{cases} 2x^2 - 1 & \text{if } -1 < x < 0 \\ \frac{x^2 - 1}{x^2 + 1} & \text{if } x \geq 0 \end{cases}
\]
We show that \( M_0 = 1 \), \( M_1 = 4 \), and \( M_2 = 4 \) for the above \( f \) defined on \((−1, \infty)\). It’s easy to verify the following:
\[
f'(x) = \begin{cases} 4x & \text{if } -1 < x < 0 \\ \frac{4x}{(x^2 + 1)^2} & \text{if } x \geq 0 \end{cases}
\]
\[
f''(x) = \begin{cases} 4 & \text{if } -1 < x < 0 \\ -\frac{12x^2 + 4}{(x^2 + 1)^3} & \text{if } x \geq 0 \end{cases}
\]
The only care one needs to take in differentiating the above \( f \) is at 0, but this is handled in a straightforward manner by comparing the right-hand and the left-hand limits at 0. Writing \( \frac{x^2 - 1}{x^2 + 1} = 1 - \frac{2}{x^2 + 1} \), it’s clear that \( M_0 = 1 \). Likewise for \( M_1 = 4 \) and \( M_2 = 4 \).

8. Rudin Ch 5, Exercise #18. Suppose \( f \) is a real function on \([a, b]\), \( n \) is a positive integer, and \( f^{(n-1)} \) exists for every \( t \in [a, b] \). Let \( \alpha, \beta, P \) be as in Taylor’s theorem (5.15). Define
\[
Q(t) = \frac{f(t) - f(\beta)}{t - \beta}
\]
for \( t \in [a, b] \), \( t \neq \beta \), differentiate
\[
f(t) - f(\beta) = (t - \beta)Q(t)
\]
n-1 times at \( t = \alpha \), and derive the following version of Taylor’s theorem:
\[
f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!} (\beta - \alpha)^n.
\]

Solution.

We’ll verify the conclusion by induction on \( n \). Note that \( P(\beta) \) depends on \( n \), as \( P(\beta) = f(\alpha) + f'(\alpha)(\beta - \alpha) + \frac{f''(\alpha)}{2!}(\beta - \alpha)^2 + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!} (\beta - \alpha)^{n-1} \).

If \( n = 1 \), then the conclusion reads \( f(\beta) = P(\beta) + Q(\alpha)(\beta - \alpha) = f(\alpha) + Q(\alpha)(\beta - \alpha) \), which just comes from the definition of \( Q(t) \) (and evaluating this at \( t = \alpha \)).
Before we proceed to the inductive step, we develop a formula for derivatives of \( f(t) - f(\beta) = (t-\beta)Q(t) \). Differentiating this once, we have \( f'(t) = Q(t) + (t-\beta)Q'(t) \), differentiating yet again, we have \( f''(t) = 2Q'(t) + (t-\beta)Q''(t) \). It’s easy to see by (yet another) induction that \( f^{(n)}(t) = nQ^{(n-1)}(t) + (t-\beta)Q^{(n)}(t) \) for \( n \geq 1 \). Plugging in \( t = \alpha \), we have \( f^{(n)}(\alpha) = nQ^{(n-1)}(\alpha) + (\alpha - \beta)Q^{(n)}(\alpha) \).

Now we resume the induction. Assuming that the conclusion holds for \( n \), we have \( f(\beta) = P(\beta) + \frac{Q^{(n)}(\alpha)}{n!}(\beta - \alpha)^n \). From here, one simply substitutes \( Q^{(n)}(\alpha) = \frac{(\alpha - \beta)^{n+1}}{(n+1)!} \) from the equation we derived in the previous paragraph.

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9. Rudin Ch 5, Exercise #26. Suppose \( f \) is differentiable on \([a, b] \), \( f(a) = 0 \), and there is a real number \( A \) such that \( |f'(x)| \leq Af(x) \) on \([a, b] \). Prove that \( f(x) = 0 \) for all \( x \in [a, b] \).

**Solution.**

Note that we may as well assume that \( A > 0 \), for if \( A \leq 0 \) then \( 0 \geq |f'(x)| \leq 0 \), or \( f'(x) = 0 \) for all \( x \in [a, b] \), in which case we have that for \( t \in [a, b] \), \( f(t) = f(t) - f(a) = f'(c)(t-a) \) for some \( c \in [a, t] \), and since \( f'(c) = 0 \) we have \( f(t) = 0 \), i.e. \( f(x) = 0 \) for all \( x \in [a, b] \).

Assuming now that \( A > 0 \), set \( \Delta = \frac{b-a}{n} \), where \( n \in \mathbb{Z}^+ \) is chosen such that \( \frac{b-a}{n} < \frac{1}{A} \) (note that we can arrange for this by the Archimedean property). For \( i \in \{0, 1, 2, \ldots, n\} \), set \( t_i = a + i\Delta \), so that \([a, b] = \bigcup_{i=1}^{n}[t_{i-1}, t_i] \).

We show by induction on \( i \) that \( f \) is 0 on \([t_{i-1}, t_i] \).

Base case is when \( i = 1 \), or showing that \( f \) is 0 on \([a, a+\Delta] \). Set \( M \) be the sup of the set \([-|f(x)| \mid x \in [a, a+\Delta] \]. Taking \( x \in [a, a+\Delta] \), we have by the Mean Value Theorem there exists \( c \in [a, x] \) such that \( |f(x)| = |f(x) - f(a)| = |f'(c)||x-a| \leq A|f(c)||x-a| \leq A \cdot \Delta \cdot M \). This being true for all \( x \in [a, a+\Delta] \), we have \( M \leq A \cdot \Delta \cdot M \). If \( M > 0 \), then dividing the inequality by \( M \), we have \( 1 \leq A \cdot \Delta \), contradicting \( \Delta < \frac{1}{A} \). Therefore \( M = 0 \) and \( f(x) = 0 \) for all \( x \in [a, a+\Delta] \).

Assume now that \( f \) is 0 on \([t_0, t_1] \cup [t_1, t_2] \cup \ldots \cup [t_{i-1}, t_i] \). We now show that \( f \) is 0 on \([t_i, t_{i+1}] \). We know that \( f(t_i) = 0 \). The same argument applies: Let \( M \) be the sup of the set \([-|f(x)| \mid x \in [t_i, t_{i+1}] \]. For any \( x \in [t_i, t_{i+1}] \), we have \( |f(x)| = |f(x) - f(t_i)| = |f'(c)||x-t_i| \leq A|f(c)| \cdot \Delta \leq A \cdot \Delta \cdot M \), for some \( c \in [t_i, x] \). This being true for all \( x \in [t_i, t_{i+1}] \), we have \( M \leq A \cdot \Delta \cdot M \). Same argument as before, we must then have \( M = 0 \). Therefore \( f = 0 \) on \([t_i, t_{i+1}] \).

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10. Rudin Ch 5, Exercise #27. Let \( \phi \) be a real function defined on a rectangle \( R \) in the plane, given by \( a \leq x \leq b, \alpha \leq y \leq \beta \). A solution of the initial-value problem

\[
y' = \phi(x, y), \quad y(a) = c \quad (\alpha \leq c \leq \beta)
\]

is, by definition, a differentiable function \( f \) on \([a, b] \) such that \( f(a) = c, \alpha \leq f(x) \leq \beta \), and

\[
f'(x) = \phi(x, f(x)) \quad (a \leq x \leq b).
\]
Prove that such a problem has at most one solution if there is a constant $A$ such that
\[ \phi(x, y_2) - \phi(x, y_1) \leq A|y_2 - y_1| \]
whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.

Solution.

Suppose $g : [a, b] \to \mathbb{R}$ is yet another solution of the above initial-value problem. Consider a function $h : [a, b] \to \mathbb{R}$ defined by $h(x) = f(x) - g(x)$ for $x \in [a, b]$. Then for $x \in [a, b], |h'(x)| = |f'(x) - g'(x)| = |\phi(x, f(x)) - \phi(x, g(x))| \leq A|f(x) - g(x)| = A|h(x)|$. Also, $h(a) = f(a) - g(a) = c - c = 0$. By problem #26 then we have $h = 0$ on $[a, b]$.

We now classify all solutions to $y' = y^{1/2}, y(0) = 0$. Certainly zero function is a solution to this differential equation. Suppose now that $f$ is a solution to the differential equation on an interval $I$ containing $0$. Certainly $f(x) \geq 0$ for all $x \in I$, so that $f'(x) = \sqrt{f(x)}$ implies that $f$ is nondecreasing on $I$. For $x \in I$ and $x \leq 0$, we must thus have $0 \leq f(x) \leq f(0) = 0$, or $f(x) = 0$. We can then just (uniquely by monotonicity of $f$) extend $f$ to all of $(-\infty, 0] \cup I$ by defining it to be $0$ on $(-\infty, 0]$, therefore we assume that our function is defined $f : (-\infty, a] \to \mathbb{R}$, where $a \geq 0$. We can assume that $a > 0$ (if $a = 0$ the monotonicity argument above already implies that zero function is the only possible solution on $(-\infty, 0]$). Zero function defined on all of $(-\infty, \infty)$ is of course a solution to the differential equation. Since $f$ is nondecreasing, if $f$ is not the zero function then $f(a) = b > 0$.

Now let $c = \sup\{x \mid f(x) = 0\} = S$. Note that $c$ exists because $S$ has $a$ as an upper bound and is nonempty because it contains $0$. Then there is a sequence $\{x_n\}_{n \geq 1}$ in $S$ such that $\lim_{n \to \infty} x_n = c$, and by continuity of $f$ we must have $f(c) = 0$. We have $c < a$. Take a point $l$ such that $c < l < a$, and note that $f(l) = s > 0$. Now let $K_n = [c + \frac{a-c}{n}, l + \frac{a-c}{n}]$ for $n \geq 2$. Then $K_n$ is compact, $f$ is defined on $K_n$ and $f$ is never zero on $K_n$. Therefore $f(K_n)$ is compact and does not contain 0. Considering the function $\phi(y) = y^{1/2}$ defined on $K_n$, we see that $\phi'(y) = \frac{1}{2} y^{-1/2}$ is defined and continuous on $f(K_n)$, so $\phi'(f(K_n))$ is compact, i.e. there is a real number $A$ (may depend on $n$) such that for all $x \in f(K_n)$, $|\phi'(x)| \leq A$. Therefore if $q, w \in f(K_n)$ then by the Mean Value Theorem, $|\phi(q) - \phi(w)| = |\phi'(c)||q - w| \leq A|q - w|$, where $c$ is between $q$ and $w$. Also $f(K_n)$ is compact and connected in $\mathbb{R}$, so it is a closed interval (Theorem 2.47, say). If we use the rectangle $R$ as in problem #27 as $K_n \times f(K_n)$, then $f'(x) = \phi(f(x))$ and $|\phi(y_1) - \phi(y_2)| \leq A|y_1 - y_2|$ for all $y_1, y_2 \in f(K_n)$, so the conditions of problem #27 are satisfied (note that $\phi(y)$ is independent of $x$ here, an “autonomous” case). Therefore the solution to this differential equation with the initial condition $f(l) = s$ is unique, and it’s readily checked that the mapping $x \mapsto \frac{(x + 2\sqrt{\pi} - l)^2}{4}$ is a solution to this initial differential equation, so we must have $f(x) = \frac{(x + 2\sqrt{\pi} - l)^2}{4}$ on $K_n$.

Now we take $x_n = c + \frac{l-c}{n}$ for $n \geq 2$, so that $x_n \in K_n$ and $f(x_n) = \frac{(x_n + 2\sqrt{\pi} - l)^2}{4}$. Note that $\lim_{n \to \infty} x_n = c$, and so by continuity of $f$ we must then have $0 = f(c) = \frac{(c + 2\sqrt{\pi} - l)^2}{4}$. $s = f(l)$, so we must have $c + 2\sqrt{f(l)} - l = 0$, or
\[ f(l) = \frac{(l-c)^2}{4} \]
Therefore
\[ f(x) = \begin{cases} 0 & \text{if } x \leq c \\ \frac{(l-c)^2}{4} & \text{if } x \geq c \end{cases} \]