HW #2 SOLUTIONS

1. Let \( f, g : [a, b] \to \mathbb{R} \) be nonnegative bounded functions.

(a) Is it necessarily true that
\[
\int_a^b f \, dx + \int_a^b g \, dx = \int_a^b (f + g) \, dx?
\]
Either prove this or show that there is a counterexample.

(b) Suppose that \( f(x) > g(x) + 1 \) for all \( x \in [a, b] \). Is it necessarily true that
\[
\int_a^b f \, dx \geq \int_a^b g \, dx?
\]
Either prove this or show that there is a counterexample.

Solution.

(a) No.

Consider the real-valued functions \( f, g \) defined on \([0, 1]\) by
\[
f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{if } x \not\in \mathbb{Q} \cap [0,1], \\ 1 & \text{if } x \not\in \mathbb{Q} \cap [0,1] \end{cases}
\]
and
\[
g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 1 & \text{if } x \not\in \mathbb{Q} \cap [0,1] \end{cases}
\]
Then \((f + g)(x) = 1\) for all \( x \in [0, 1]\).

In computing \( \int_0^1 f \, dx \), as usual we first take an arbitrary partition \( P \) of \([0, 1]\), say described by \( 0 = x_0 < x_1 < \cdots < x_n = 1 \). Since \([0, 1] - \mathbb{Q} \cap [0, 1]\) is dense in \([0, 1]\), \( \inf_{y \in [x_{i-1}, x_i]} f(y) = 0 \) for \( i \geq 1 \) (here we need the fact that \( x_{i-1} < x_i \), a strict inequality). Therefore \( L(P, f) = 0 \). This being true for an arbitrary partition \( P \) of \([0, 1]\), we have then that \( \int_0^1 f \, dx = \sup_P L(P, f) = 0 \).

The computation for \( \int_0^1 g \, dx = 0 \) is similar, using this time the fact that \( \mathbb{Q} \cap [0, 1]\) is dense in \([0, 1]\).

For the computation of \( \int_0^1 (f + g) \, dx \), note that it equals \( \int_0^1 1 \, dx \) because \((f + g)(x) = 1\) for all \( x \in [0, 1]\) and the function mapping \( x \mapsto 1 \) for all \( x \in [0, 1]\) is a continuous function, hence Riemann integrable. One can use the Fundamental Theorem of Calculus now to check that the value of this integral is then \( 1 \). Note that one can also compute this directly from the definition: Take any partition \( P \) of \([0, 1]\), again described by \( 0 = x_0 < x_1 < \cdots < x_n = 1 \). Then in each subinterval \([x_{i-1}, x_i]\), our function \( f \) takes on the value \( 1 \) constantly. That is, \( \inf_{y \in [x_{i-1}, x_i]} f(y) = 1 \) for each \( i \geq 1 \). Therefore \( L(P, f) = \sum_{i=1}^n (x_i - x_{i-1}) = x_n - x_0 = 1 - 0 = 1 \). Taking supremum over the partitions, we reach \( 1 = \overline{\int_0^1 (f + g)} \, dx \).

(b) No.

Consider the functions \( f, g \) defined on \([0, 1]\) by
\[
f(x) = \begin{cases} 5 & \text{if } x \in \mathbb{Q}, \\ 2 & \text{if } x \not\in \mathbb{Q} \end{cases}
\quad g(x) = \begin{cases} 3 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \not\in \mathbb{Q} \end{cases}
\]
We then have \( f(x) > g(x) + 1 \) for all \( x \in [0, 1]\). We compute the lower and the upper integrals of \( f, g \), respectively, separately:
\[
\int_0^1 f \, dx. \quad \text{Again take any partition } P \text{ of } [0, 1], \text{ described by } 0 = x_0 < x_1 < \cdots < x_n = 1. \text{ Because } [0, 1] - \mathbb{Q} \text{ is dense in } [0, 1], \text{ } \inf_{y \in [x_{i-1}, x_i]} f(y) = 2 \text{ for } i \geq 1 \text{ (again we use here that } x_i > x_{i-1}\text{)}. \text{ Therefore } L(P, f) = \sum_{i=1}^n 2(x_i - x_{i-1}) = 2(x_n - x_0) = 2. \text{ So } \int_0^1 f \, dx = 2.
\]
\[ \int_0^1 g \, dx. \] Take a partition \( P \) of \([0, 1]\), \( 0 = x_0 < x_1 < \cdots < x_n = 1 \). \( \sup_{y \in [x_{i-1}, x_i]} g(y) = 3 \) if \( x_{i-1} < x_i \) (\( i \geq 1 \)) because \( \mathbb{Q} \cap [0, 1] \) is dense in \([0, 1]\). Therefore \( U(P, f) = 3 \) in this case, taking infimum over all the partitions, we have \( \int_0^1 g \, dx = 3 \).

\[ \int_0^1 f \, dx \text{ and } \int_0^1 f \, dx. \]

Solution.

We compute \( \int_0^1 f \, dx \). The computation for \( \int_0^1 f \, dx \) is similar and uses \( X \cap I \neq I \). Take a partition \( P \) of \([0, 1]\), defined again by \( 0 = x_0 < x_1 < \cdots < x_n = 1 \). Then \( U(P, f) = \sum_{i=1}^n \sup_{y \in [x_{i-1}, x_i]} f(y) \cdot (x_i - x_{i-1}) \). We now compute, for \( i \geq 1 \), \( \sup_{y \in [x_{i-1}, x_i]} f(y) \). For any \( y \in [x_{i-1}, x_i], f(y) \) is either \( y \) or \( -y \), i.e. \( |f(y)| = |y| \leq x_i \).

Therefore \( x_i \) is an upper bound of the set \( S = \{ f(y) | y \in [x_{i-1}, x_i] \} \). If \( s < x_i \), then we consider the interval \( I = (s, x_i) \), which is nonempty. By the hypothesis on \( X \), we have \( X \cap I \neq \emptyset \), so we may take \( y \in X \cap I \). Then \( f(y) = y > s \), so \( s \) is not an upper bound of \( S \). Therefore \( x_i = \sup_{y \in [x_{i-1}, x_i]} f(y) \), so \( U(P, f) = \sum_{i=1}^n x_i(x_i - x_{i-1}) \).

Denoting \( g : [0, 1] \to \mathbb{R} \) for the function defined by \( g(x) = x \) for all \( x \in [0, 1] \), then we have \( U(P, f) = U(P, g) \) if \( P \) is a partition of \([0, 1]\), so we have \( \int_0^1 f \, dx = \int_0^1 g \, dx = \int_0^1 x \, dx \) where the last equality follows because \( g \) is a continuous function on \([0, 1]\), hence Riemann integrable. By the fundamental theorem of calculus then, \( \int_0^1 x \, dx = \frac{1}{2} \).

Remark. Please refer to Piazza post in regards to computing this without resorting to the fundamental theorem of calculus.

In a similar manner, one computes that \( \int_0^1 f \, dx = -\frac{1}{2} \).

3. Let \( a < b < c \). Define the function \( \alpha : [-1, 1] \to \mathbb{R} \) by

\[ \alpha(x) = \begin{cases} 
    a & \text{if } x < 0, \\
    b & \text{if } x = 0, \\
    c & \text{if } x > 0. 
\end{cases} \]

Let \( f : [-1, 1] \to \mathbb{R} \) be a bounded function. Prove that \( f \in \mathcal{R}(\alpha) \) if and only if \( f \) is continuous at 0.

Solution.

Suppose first that \( f \in \mathcal{R}(\alpha) \). Take any \( \epsilon > 0 \). Then there exists a partition \( P \) of \([-1, 1]\) such that \( U(P, f) - L(P, f) < \epsilon \cdot \min(b - a, c - b) \). By including 0 in \( P \) if necessary, we may as well assume that 0 \( \in P \) (recall that the inequality \( U(P, f) - L(P, f) < \epsilon \) does not change when we refine \( P \)). Let \( P = \{ x_0, x_1, \ldots, x_k, 0, y_1, \ldots, y_r \} \)
where $-1 = x_0 < x_1 < \cdots < x_k < 0 < y_1 < \cdots < y_r = 1$. Then the only terms in the difference of the sum \( U(P, f) - L(P, f) \) that survive are precisely those corresponding to the subintervals \([x_k, 0]\) and \([0, y_1]\). Therefore \( U(P, f) - L(P, f) = (\sup_{y \in [x_k, 0]} f(y) - \inf_{y \in [x_k, 0]} f(y))(b - a) + (\sup_{y \in [0, y_1]} f(y) - \inf_{y \in [0, y_1]} f(y))(c - b) < \epsilon \cdot \min(b - a, c - b) \).

Therefore \( (\sup_{y \in [x_k, 0]} f(y) - \inf_{y \in [x_k, 0]} f(y))(b - a) < \epsilon \cdot \min(b - a, c - b) \) in particular, and hence \( \sup_{y \in [x_k, 0]} f(y) - \inf_{y \in [x_k, 0]} f(y) < \epsilon \). Likewise, \( \sup_{y \in [0, y_1]} f(y) - \inf_{y \in [0, y_1]} f(y) < \epsilon \). Now \( 0 \in (x_k, y_1) \), the latter is an open neighborhood of 0, and if \( a \in (x_k, y_1) \) then \( |f(a) - f(0)| \leq \max(\sup_{y \in [x_k, 0]} f(y) - \inf_{y \in [x_k, 0]} f(y), \sup_{y \in [0, y_1]} f(y) - \inf_{y \in [0, y_1]} f(y)) < \epsilon \). Taking \( \delta = \min(x_k, y_1) \), we see that if \( |a| < \delta \) then \( |f(a) - f(0)| < \epsilon \). \( \epsilon > 0 \) was arbitrarily chosen, so \( f \) is continuous at 0.

Suppose now that \( f \) is continuous at 0. Take any \( \epsilon > 0 \). Let \( \epsilon' = \frac{\epsilon}{6 \max(b - a, c - b)} \). Since \( f \) is continuous at 0, there exists \( \delta > 0 \) such that if \( |x - 0| < \delta \) then \( |f(x) - f(0)| < \epsilon' \). Therefore \( f(0) - \epsilon' < f(x) < f(0) + \epsilon' \). Therefore \( f(0) - \epsilon' \leq \sup_{y \in [-\frac{\delta}{2}, 0]} f(y) \leq f(0) \) by \( \epsilon' \), i.e. \( \sup_{y \in [-\frac{\delta}{2}, 0]} f(y) \leq f(0) + \epsilon' \). Similarly, \( \inf_{y \in [0, \frac{\delta}{2}]} f(y) \leq 2\epsilon' \). Taking for our partition \( P \) of \([-1, 1]\) as \( P = \{-1, -\frac{\delta}{2}, 0, \frac{\delta}{2}, 1\} \), \( U(P, f) - L(P, f) = (\sup_{y \in [-\frac{\delta}{2}, 0]} f(y) - \inf_{y \in [-\frac{\delta}{2}, 0]} f(y))(b - a) + (\sup_{y \in [0, \frac{\delta}{2}]} f(y) - \inf_{y \in [0, \frac{\delta}{2}]} f(y))(c - b) < \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon \). This shows that \( f \in \mathcal{R}(\alpha) \).

4. Let \( \alpha : [a, b] \to \mathbb{R} \) be a nondecreasing function.

(a) Let \( x_0 \in (a, b) \) and suppose that \( \alpha \) is continuous at \( x_0 \). Define \( f : [a, b] \to \mathbb{R} \) by \( f(x_0) = 1 \) and \( f(x) = 0 \) for \( x \neq x_0 \). Prove that \( f \in \mathcal{R}(\alpha) \) and that \( \int f \, d\alpha = 0 \).

(b) Let \( a < x_1 < x_2 < \cdots < x_k < b \) and suppose that \( \alpha \) is continuous at each of \( x_1, x_2, \ldots, x_k \). Let \( c_i \in \mathbb{R} \) for \( 1 \leq i \leq k \). Define \( g(x_i) = c_i \) for \( 1 \leq i \leq k \) and \( g(x) = 0 \) if \( x \notin \{x_1, x_2, \ldots, x_k\} \). Prove that \( g \in \mathcal{R}(\alpha) \) and that \( \int g \, d\alpha = 0 \).

Solution.

Suppose (a) holds. If \( g \) satisfies the conditions in (b), then \( g = c_1 f_1 + \cdots + c_k f_k \) where \( f_i : [a, b] \to \mathbb{R} \) is a function such that \( f_i(x_i) = 1 \) and 0 elsewhere. Since a linear combination of Riemann-Stieltjes integrable functions are again Riemann-Stieltjes integrable, \( g \in \mathcal{R}(\alpha) \) and we also have the linearity \( \int_a^b g \, d\alpha = c_1 \int_a^b f_1 \, d\alpha + c_2 \int_a^b f_2 \, d\alpha + \cdots + c_k \int_a^b f_k \, d\alpha = 0 \).

We need only solve (a) now then. Take any \( \epsilon > 0 \). Since \( \alpha \) is continuous at \( x_0 \), there exists \( \delta > 0 \) such that \( |\alpha(y) - \alpha(x_0)| < \epsilon \) if \( |y - x_0| < \delta \). Take a partition \( P = \{a, x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}, b\} \) (we can arrange \( \delta \) small so that \( a < x_0 - \frac{\delta}{2} < x_0 + \frac{\delta}{2} < b \)). Then \( L(P, f) = 0 \) clearly and \( U(P, f) = 1 \cdot (\alpha(x_0 + \frac{\delta}{2}) - \alpha(x_0 - \frac{\delta}{2})) = \alpha(x_0 + \frac{\delta}{2}) - \alpha(x_0 - \frac{\delta}{2}) < \epsilon \). This shows that \( f \) is in \( \mathcal{R}(\alpha) \) but also \( \int_a^b f \, d\alpha = 0 \).

5. (a) Define \( f : [0, 2] \to \mathbb{R} \) by \( f(1/n) = 1 \) for \( n \in \mathbb{Z}^+ \) and \( f(x) = 0 \) for \( x \in [0, 2] - \bigcup_{n=1}^{\infty} \{1/n\} \). Prove or disprove that \( f \in \mathcal{R} \). If \( f \in \mathcal{R} \), find \( \int_0^2 f \, dx \).

(b) Let \( K \subset [a, b] \) be a set with the property that for any \( \epsilon > 0 \) there exists a finite union \( I \) of intervals
containing \( K \) with the total length of \( I \) being \( \leq \epsilon \). Let \( f: [a, b] \to \mathbb{R} \) be a bounded function which is continuous on \([a, b] - K\). Prove that \( f \in \mathcal{R} \).

**Solution.**

(a). Take any \( \epsilon > 0 \). Let \( S = \{ n \in \mathbb{Z}^+ \mid \frac{1}{n} > \frac{\epsilon}{2} \} = \{1, 2, \ldots, N\} \). Note that \( S \) is a finite set because the number of positive integers \( n \) such that \( n < \frac{2}{\epsilon} \) is finite. Let \( \delta \) be the minimum of the list of numbers \( \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N} - \frac{1}{N} - \frac{\epsilon}{2} \). Take \( r = \min(\frac{\delta}{2N}, \frac{\epsilon}{2}) \), a positive number. Let \( P \) be the partition of \([0, 2] , P = \{0, \frac{1}{2}\} \cup \bigcup_{n=1}^{N} \{ \frac{1}{n} - r, \frac{1}{n} + r \} \). Then the subintervals created by this partition are \([0, \frac{1}{2}], [\frac{1}{n} - r, \frac{1}{n} + r] \)'s for \( 1 \leq n \leq N \), and everything in between the endpoints. Note that the device of using \( \delta \) above is to ensure that the interiors of these subintervals are disjoint (there’s no “overlap”).

Since there is some positive integer \( k \) such that \( 1/k < \frac{\epsilon}{2} \) and there is also a number \( x \in (0, \frac{1}{2}) \) not of the form \( 1/n \) for any positive integer \( n \) (it’s easy to show this - there are uncountably many elements in \((0, \frac{1}{2})\), for example), we have \( \sup_{y \in [0, x/2]} f(y) = 1 \) whereas \( \inf_{y \in [0, \epsilon/3]} f(y) = 0 \). Likewise, In subintervals of the form \([1/n - r, 1/n + r] \) with \( 1 \leq n \leq N \), sup of \( f \) is 1 and inf of \( f \) is 0. In all other intervals, \( f \) is 0. Therefore \( U(P, f) = \frac{\epsilon}{2} + \sum_{n=1}^{N} 2r = \frac{\epsilon}{2} + 2Nr \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \), and \( L(P, f) = 0 \). Since \( L(P, f) \leq U(P, f) \), we have \( U(P, f) - L(P, f) < \epsilon \) and so \( f \in \mathcal{R} \).

The above argument shows that for all \( \epsilon > 0 \), there is a partition \( P \) of \([0, 2] \) such that \( U(P, f) < \epsilon \) and \( L(P, f) = 0 \). Therefore \( \int_{0}^{2} f \, dx = 0 \).

(b). Let \( A = \sup_{y \in [a, b]} |f(y)| \). Take \( \epsilon > 0 \). By the hypothesis on \( K \), there exists a finite union of intervals, \( I = (a_1, b_1) \cup \cdots \cup (a_n, b_n) \), such that \( K \subset I \) and \( \sum_{i=1}^{n} (b_i - a_i) < \frac{\epsilon}{2A} \). Note that a union of a pair of open intervals is again an open interval, so we may as well assume that the intervals \((a_1, b_1), \ldots, (a_n, b_n)\) are pair-wise disjoint. Then \([a, b] - I \) is a finite disjoint union of closed intervals, say \([a, b] - I = [c_1, d_1] \cup \cdots \cup [c_r, d_r] \) (note that this includes the case when one of the \((a_i, b_i)\)'s contains an endpoint, either \( a \) or \( b \), or even the case when \( c_i = d_i \) for some \( i \) so that \([c_i, d_i]\) is a single-ton). Since \([a, b] - I \subset [a, b] - K \), \( f \) is continuous on \([a, b] - I \), hence on all of \([c_i, d_i]\)'s.

Therefore \( f \) is Riemann-integrable in each of these closed subintervals, and so there exist partitions \( P_1, P_2, \ldots, P_r \) such that \( U(P_i, f|_{[c_i, d_i]}) - L(P_i, f|_{[c_i, d_i]}) < \frac{\epsilon}{2} \) for \( 1 \leq i \leq r \). Take the partition \( P = P_1 \cup \cdots \cup P_r \cup \{a, b\} \). Then, \( U(P, f) - L(P, f) \leq \sum_{i=1}^{r} (U(P_i, f) - L(P_i, f)) + A \cdot \text{total length of } I < \epsilon/2 + \epsilon/2 = \epsilon \). So \( f \) is Riemann-integrable on \([a, b] \).

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6. Exercise #10 on p. 139 (Principles by Rudin). Let \( p \) and \( q \) be positive real numbers such that

\[
\frac{1}{p} + \frac{1}{q} = 1.
\]

Prove the following statements.

(a) If \( u \geq 0 \) and \( v \geq 0 \), then

\[
uv \leq \frac{u^p}{p} + \frac{v^q}{q}.
\]

Equality holds if and only if \( u^p = v^q \).

(b) If \( f \in \mathcal{R}(\alpha) \), \( g \in \mathcal{R}(\alpha) \), \( f \geq 0 \), \( g \geq 0 \), and

\[
\int_{a}^{b} f^p \, d\alpha = 1 = \int_{a}^{b} g^q \, d\alpha,
\]

then \( \int_{a}^{b} f \cdot g \, d\alpha \).
then
\[ \int_a^b fg \, d\alpha \leq 1. \]

(c) If \( f \) and \( g \) are complex functions in \( \mathcal{R}(\alpha) \), then
\[ \left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}. \]

This is Hölder’s inequality.

(d) Show that Hölder’s inequality is also true for the “improper” integrals described in Exercises 7 and 8.

Solution.

(a) We can make use of the convexity of the function \( f(x) = e^x \), which satisfies \( f''(x) = e^x > 0 \), so that \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \) if \( t \in (0,1) \) and \( x < y \). Plugging \( t = \frac{1}{p} \), so that \( 1 - t = \frac{1}{q} \) and setting \( x, y \) such that \( u^p = e^x \) and \( v^q = e^y \) assuming that \( u \neq 0 \) and \( v \neq 0 \) does the job. In the case of \( u = v = 0 \), the inequality clearly holds.

We resort to a more direct method. If \( u = 0 \) or \( v = 0 \) then the statement follows trivially. So assume that \( u > 0 \) and \( v > 0 \). Dividing through by \( v^q \), the inequality is equivalent to \( \frac{u}{v^q} \leq \frac{1}{p} + \frac{1}{q} \). Letting \( t = \frac{u}{v^q} \), note that \( t^p = \frac{u^p}{v^q} \) making use of the relation \( p(q - 1) = q \). So it suffices to prove that the function \( f(t) = t - \frac{1}{p} t^p \) defined on \((0, \infty)\) reaches maximum value of \( \frac{1}{q} \) and that this happens precisely when \( t = 1 \), i.e. \( u = v^{q-1} \) or \( u^p = v^q \). But \( f'(t) = 1 - t^{p-1} \), which is \( < 0 \) if \( t > 1 \) and \( > 0 \) if \( 0 < t < 1 \) (note that \( p > 1 \) from the relation \( \frac{1}{p} + \frac{1}{q} = 1 \)). Therefore \( f \) is increasing on \((0,1)\) and decreasing on \((1, \infty)\), i.e. the maximum value is reached at \( t = 1 \) and the maximum value is \( f(1) = 1 - \frac{1}{p} = \frac{1}{q} \).

(b) By part (a), we have \( fg \leq \frac{1}{p} f^p + \frac{1}{q} g^q \). Therefore \( \int_a^b fg \, d\alpha \leq \frac{1}{p} \int_a^b f^p \, d\alpha + \frac{1}{q} \int_a^b g^q \, d\alpha = \frac{1}{p} + \frac{1}{q} = 1 \).

(c) In the case when \( \int_a^b |f|^p \, d\alpha > 0 \) and \( \int_a^b |g|^q \, d\alpha > 0 \), one simply divides through the inequality by right side of the equation to reduce to the case in part (b) (replacing \( f \) by \( f/\left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \) and likewise for \( g \)) and also making use of proposition \( \left| \int_a^b f \, h \, d\alpha \right| \leq \int_a^b |h| \, d\alpha \).

Suppose without a loss of generality that \( \int_a^b |f|^p \, d\alpha = 0 \). We need to show that \( \int_a^b |fg| \, d\alpha = 0 \). Note that there is no problem in writing \( \int_a^b fg \, d\alpha \) because product of functions in \( \mathcal{R}(\alpha) \) is again in \( \mathcal{R}(\alpha) \). If \( M = \sup_y |g(y)| \), then \( \int_a^b |fg| \, d\alpha \leq M \int_a^b |f| \, d\alpha \), so it suffices to check that \( \int_a^b |f| \, d\alpha = 0 \) if \( \int_a^b |f|^p \, d\alpha = 0 \) (\( p > 1 \) here, though the implication actually also holds for \( p > 0 \) in general). Suppose, on the contrary, that \( \int_a^b |f| \, d\alpha > 0 \). Then there is a partition \( P \) of \([a, b]\) such that \( L(P, f) > 0 \), so there is a subinterval \([c, d]\) of \([a, b]\) (\( a \leq c \leq d \leq b \)) such that \( \inf_{y \in [c, d]} |f(y)| : (\alpha(d) - \alpha(c)) > 0 \). Let \( m = \inf_{y \in [c, d]} |f(y)| \). So \( m(\alpha(d) - \alpha(c)) > 0 \). Therefore \( 0 < m^p (\alpha(d) - \alpha(c)) = \int_c^d m^p \, d\alpha \leq \int_c^d |f|^p \, d\alpha \leq \int_a^b |f|^p \, d\alpha = 0 \), contradiction. Therefore \( \int_a^b |f| \, d\alpha = 0 \) and we’re done.

(d) This just follows by taking the limit of the inequality in (c) as endpoints tend to \( \infty \).
7. Suppose \( f \in \mathcal{R}(\alpha) \) on \([a, b]\) (\(a, b \in \mathbb{R}\)), \(p \geq 1\), and \( \epsilon > 0 \). Prove that there exists a continuous function \( g \) on \([a, b]\) such that \( \|f - g\|_p < \epsilon \).

Solution.

Take any \( \epsilon > 0 \) and choose a partition \( P \) of \([a, b]\), say \( a = x_0 < x_1 < \cdots < x_n = b \) such that \( U(P, f) - L(P, f) < \epsilon \), or \( \sum_{i=1}^{n}(M_i - m_i)(\alpha(x_i) - \alpha(x_{i-1})) < \epsilon \), where \( M_i = \sup_{y \in [x_{i-1}, x_i]} f(y) \) and \( m_i = \inf_{y \in [x_{i-1}, x_i]} f(y) \). Take \( g \) to be a piece-wise linear function where on \([x_{i-1}, x_i]\) we define it to be \( g(t) = \frac{x_i - t}{x_i - x_{i-1}} f(x_{i-1}) + \frac{t - x_{i-1}}{x_i - x_{i-1}} f(x_i) \). Since \( |f - g|^p \geq 0 \), \( L(P, |f - g|^p) \geq 0 \). As for the upper sum, we have \( U(P, |f - g|^p) = \sum_{i=1}^{n} \sup_{y \in [x_{i-1}, x_i]} |f(y) - g(y)|^p(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^{n} \sup_{y \in [x_{i-1}, x_i]} |f(y) - g(y)| \cdot (\sup_{y \in [x_{i-1}, x_i]} |f(y) - g(y)|)^{p-1} \cdot (\alpha(x_i) - \alpha(x_{i-1})) \).

Note that we have used \( \sup_{y \in [x_{i-1}, x_i]} |f(y) - g(y)|^{p-1} = (\sup_{y \in [x_{i-1}, x_i]} |f(y) - g(y)|)^{p-1} \) (you can verify this by using the continuity of the function \( x \mapsto x^{p-1} \), defined everywhere on \( \mathbb{R} \) including 0, since \( p \geq 1 \). From the way \( g \) is defined, it’s clear that on \([x_{i-1}, x_i]\) (\(i \geq 1\)), we have \( m_i \leq f, g \leq M_i \). Therefore if \( y \in [x_{i-1}, x_i] \) then \( |f(y) - g(y)| \leq M_i - m_i \). That is, \( \sup_{y \in [x_{i-1}, x_i]} |f(y) - g(y)| \leq M_i - m_i \). Also, \( \sup_{y \in [x_{i-1}, x_i]} |f(y) - g(y)|^{p-1} \leq (2M)^{p-1} \) if \( M = \sup_{y \in [a, b]} |f(y)| \) is the bound on \(|f|\) on the entire interval \([a, b]\). Therefore \( U(P, |f - g|^p) \leq (2M)^{p-1} \sum_{i=1}^{n} (M_i - m_i)(\alpha(x_i) - \alpha(x_{i-1})) \leq (2M)^{p-1} \epsilon, \) which implies that \( \|f - g\|_2 \leq \left( \int_a^b |f - g|^p \, d\alpha \right)^{1/p} \leq U(P, |f - g|^p)^{1/p} \leq (2M)^{1-\frac{1}{p}} \epsilon. \) Since \( \epsilon > 0 \) was arbitrary, we’re done.