HW #3 SOLUTIONS

1. Do Exercise #1 on p. 165 of Rudin. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Solution.

We need to check that if \( f : E \to \mathbb{C} \) is a function on a set \( E \) and \( f_1, f_2, \ldots \) is a sequence of functions from \( E \) to \( \mathbb{C} \) such that \( f_n \to f \) uniformly on \( E \) and there exist a sequence \( M_1, M_2, \ldots \) of positive reals such that \( |f_n(x)| \leq M_n \) for all \( x \in E \) and \( n \in \mathbb{Z}^+ \) then there exists \( M > 0 \) such that \( |f_n(x)| \leq M \) for all \( n \in \mathbb{Z}^+ \) and \( x \in E \).

Since \( \{f_n\} \) converges uniformly, it’s in particular uniformly Cauchy and there exists \( N \in \mathbb{Z}^+ \) such that if \( n, m \geq N \) then \( \sup_{x \in E} |f_n(x) - f_m(x)| < 1 \). So if \( n \geq N \), \( x \in E \), then \( |f_n(x)| \leq |f_n(x) - f_N(x)| + |f_N(x)| < 1 + M_N \). Taking \( M = \max(M_1, M_2, \ldots, M_N) + 1 \), we have \( \sup_{x \in E} |f_n(x)| \leq M \) for all \( n \in \mathbb{Z}^+ \).

2. Let \( \{a_n\}_{n=1}^{\infty} \) be a strictly decreasing sequence of real numbers in \((0, \frac{1}{2})\) with \( \lim_{n \to \infty} a_n = 0 \). Let \( \{b_n\}_{n=1}^{\infty} \) be a sequence of positive real numbers. Define continuous functions \( f_n : [0, 1] \to \mathbb{R} \) so that \( f_n(0) = f_n(2a_n) = f_n(1) = 0 \) and \( f_n(a_n) = b_n \) and \( f_n \) is linear in each of the intervals \([0, a_n], [a_n, 2a_n], \) and \([2a_n, 1]\).

(a) Prove that \( f_n \) converges pointwise to the zero function.
(b) Find a necessary and sufficient condition on the sequence \( \{b_n\}_{n=1}^{\infty} \) for the convergence of \( f_n \) to the zero function to be uniform.

Solution.

(a) Explicitly, for each positive integer \( n \), \( f_n : [0, 1] \to \mathbb{R} \) is defined by \( f_n(x) = \begin{cases} \frac{b_n}{a_n}x & \text{if } x \in [0, a_n] \\ -\frac{b_n}{a_n}x + 2b_n & \text{if } x \in [a_n, 2a_n] \\ 0 & \text{if } x \in [2a_n, 1] \end{cases} \).

Since \( f_n(0) = 0 \) for all \( n \in \mathbb{Z}^+ \), \( \lim_{n \to \infty} f_n(0) = 0 \) exists. If \( x \in (0, 1) \), then because \( \lim_{n \to \infty} a_n = 0 \), we also have \( \lim_{n \to \infty} 2a_n = 0 \) and there exists \( N \in \mathbb{Z}^+ \) such that if \( n \geq N \) then \( 2a_n < x \), so that \( f_n(x) = 0 \). That is, \( \lim_{n \to \infty} f_n(x) = 0 \) in either case, \( \lim_{n \to \infty} f_n(x) = 0 \), so \( f_n \to 0 \) point-wise on \([0, 1]\).

(b) We claim that \( f_n \to 0 \) uniformly if and only if \( \lim_{n \to \infty} b_n = 0 \). It suffices to check that \( \sup_{x \in [0,1]} |f_n(x)| = b_n \), since then \( f_n \to 0 \) uniformly if and only if \( \lim_{n \to \infty} \left( \sup_{x \in [0,1]} |f_n(x)| \right) = 0 \) if and only if \( \lim_{n \to \infty} b_n = 0 \).

Since \( |f_n(x)| \leq b_n \) for all \( x \in [0,1] \), \( \sup_{x \in [0,1]} |f_n(x)| \leq b_n \) certainly. Suppose \( 0 \leq l < b_n \). Then \( \frac{a_l}{b_n} < \frac{a_n b_n}{b_n} = a_n \), so that there exists \( 0 \leq \frac{a_l}{b_n} < x < a_n \), in particular \( x \in [0, a_n] \), and so \( |f_n(x)| = f_n(x) = \frac{b_n}{a_n}x > l \), or that \( l \) is not an upper bound of the set \( \{|f_n(x)| | x \in [0,1]\} \). Therefore \( \sup_{x \in [0,1]} |f_n(x)| = b_n \).

3. (a) Construct a sequence \( \{f_n\} \) of functions on \( \mathbb{R} \) for which \( \{f_n\} \) converges uniformly on \( \mathbb{R} \) but \( \{f_n^2\} \) does
not converge uniformly on \( \mathbb{R} \).

(b) Does there exist a sequence \( \{g_n\} \) of functions on \([0,1]\) for which \( \{g_n\} \) converges uniformly on \([0,1]\) but \( \{g_n^2\} \) does not converge uniformly on \([0,1]\)? Prove or disprove.

Solution.

(a) Take \( f_n(x) = x + \frac{1}{n} \) and \( f(x) = x \), all defined on \( \mathbb{R} \). Then \( f_n \to f \) uniformly since \( \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} \frac{1}{n} = \frac{1}{n} \), with the latter tending to 0 as \( n \to \infty \). However, for \( x \in \mathbb{R} \), \( f_n(x)^2 - f(x)^2 = (x + \frac{1}{n})^2 - x^2 = \frac{2x}{n} + \frac{1}{n^2} \), so that \( \sup_{x \in \mathbb{R}} |f_n(x)^2 - f(x)^2| = \sup_{x \in \mathbb{R}} \left( \frac{2x}{n} + \frac{1}{n^2} \right) = \infty \).

(b) Take \( g_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{x}{n} & \text{if } x \in (0,1] \end{cases} \). Then \( \lim_{n \to \infty} g_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{x}{n} & \text{if } x \in (0,1] \end{cases} \) and the convergence \( g_n \to g \) is uniform because \( \sup_{x \in [0,1]} |g_n(x) - g(x)| = \frac{1}{n} \). However, if \( x \in (0,1] \), then \(|g_n(x)^2 - g(x)^2| = \frac{2}{n^2} + \frac{1}{n^2} \), so that \( \sup_{x \in [0,1]} |g_n(x)^2 - g(x)^2| = \infty \), implying that \( g_n^2 \) does not converge to \( g^2 \) uniformly.

4. Let \( \{f_n\} \) and \( \{g_n\} \) be uniformly converging sequences of functions on a set \( E \).

(a) Prove that \( \{f_n - g_n\} \) converges uniformly on \( E \).

(b) Assume that \( \{f_n\} \) is a sequence of bounded functions. Prove that \( \{f_n^2\} \) converges uniformly on \( E \).

(c) Suppose that \( f_n > 0 \) for each \( n \). Suppose that \( \{1/f_n\} \) converges to a function \( h \) on \( E \). Does \( \{1/f_n\} \) necessarily converge uniformly to \( h \)?

Solution.

(a) Suppose \( f_n \to f \) and \( g_n \to g \) uniformly on \( E \). Given \( \epsilon > 0 \), there exist \( N_1, N_2 \) positive integers such that if \( n \geq N_1 \) then \( \sup_{x \in E} |f_n(x) - f(x)| < \epsilon/2 \) and if \( n \geq N_2 \) then \( \sup_{x \in E} |g_n(x) - g(x)| < \epsilon/2 \). Taking \( N = \max(N_1, N_2) \), we have that if \( n \geq N \) and \( x \in E \) then \( |(f-g)(x) - (f_n-g_n)(x)| = |f(x)-f_n(x) + g_n(x) - g(x)| \leq |f(x) - f_n(x)| + |g_n(x) - g(x)| \leq \sup_{y \in E} |f(y)| + \sup_{x \in E} |g(t)| < \epsilon/2 + \epsilon/2 = \epsilon \). This being true for all \( x \in E \), \( \sup_{x \in E} |(f-g)(x) - (f_n-g_n)(x)| \leq \epsilon \), so \( f_n - g_n \) converges uniformly to \( f - g \).

(b) By problem \#1, \( \{f_n\} \) must be uniformly bounded, i.e. there exists \( M > 0 \) such that \( \sup_{x \in E} |f_n(x)| \leq M \) for all \( n \in \mathbb{Z}^+ \). Therefore for \( x \in E \), we have \( \lim_{n \to \infty} |f_n(x)| \leq M \), or \( |f(x)| \leq M \) as well.

Take \( \epsilon > 0 \) now. Then there exists \( N \in \mathbb{Z}^+ \) such that if \( n \geq N \) then \( \sup_{x \in E} |f_n(x) - f(x)| < \epsilon/(2M) \), so that if \( x \in E \) then \( |f_n(x)^2 - f(x)^2| = |f_n(x) - f(x)||f_n(x) + f(x)| < \epsilon/(2M) \cdot 2M = \epsilon \), i.e. \( f_n^2 \to f^2 \) uniformly.

(c) No. Take \( E = (0,1) \subset \mathbb{R} \) and \( f_n : E \to \mathbb{R} \) for each positive integer \( n \), by \( f_n(x) = x + \frac{1}{n} \). Also, define \( f : E \to \mathbb{R} \) by \( f(x) = x \) if \( x \in (0,1) \). Then \( f_n \to f \) uniformly on \( E \), as \( \sup_{x \in (0,1)} |f_n(x) - f(x)| = \sup_{x \in (0,1)} \frac{1}{n} = \frac{1}{n} \), which tends to 0 as \( n \to \infty \). Since \( f_n > 0 \) and \( f > 0 \) on \( E = (0,1) \), we have \( 1/f_n \to 1/f = h \) point-wise. However, for \( x \in E \) and \( n \in \mathbb{Z}^+ \), \( |1/f_n(x) - 1/f(x)| = \left| \frac{1}{x + \frac{1}{n}} - \frac{1}{x} \right| = \frac{1}{n} \cdot \frac{1}{x(x + \frac{1}{n})} \), which tends to \( \infty \) as \( x \to 0^+ \).

5. (a) Let \( f_n : [1,\infty) \to \mathbb{R} \) be a sequence of functions satisfying \( |f_n(x)| \leq a_n |x| \) for \( x \in [1,\infty) \) and \( n \geq 1 \), where \( \{a_n\} \) is a sequence of positive real numbers. Suppose that \( f_n \) converges uniformly to a function \( g \) on \([1,\infty) \). Is it
necessarily true that there exists a number \( C \) such that \( |g(x)| \leq C|x| \) for \( x \in [1, \infty) \)?

(b) Let \( f_n : [-1, 1] \to \mathbb{R} \) be a sequence of functions satisfying \( |f_n(x)| \leq a_n|x| \) for \( x \in [-1, 1] \) and \( n \geq 1 \), where \( \{a_n\} \) is a sequence of positive real numbers. Suppose that \( f_n \) converges uniformly to a function \( g \) on \([−1, 1]\). Is it necessarily true that there exists a number \( C \) such that \( |g(x)| \leq C|x| \) for \( x \in [-1, 1] \)?

**Solution.**

(a) Yes it’s true. Since \( f_n \to g \) uniformly, there exists \( N \in \mathbb{Z}^+ \) such that if \( n \geq N \) then \( \sup_{x \in [1, \infty)} |f_n(x) - g(x)| < 1 \). Then for \( x \in [1, \infty) \), we have

\[
|g(x)| \leq |g(x) - f_N(x)| + |f_N(x)| < 1 + a_N|x| \leq |x| + a_N|x| = (1 + a_N)|x|.
\]

So we may take \( C = 1 + a_N \).

(b) No. Define for each positive integer \( n \), \( f_n : [-1, 1] \to \mathbb{R} \) by

\[
f_n(x) = \begin{cases} 0 & \text{if } x \in [-1, 0] \\ nx & \text{if } x \in (0, 1] \\ \frac{x}{\sqrt{x}} & \text{if } x \in (1, 1] \end{cases}
\]

\( \mathbb{R} \) be defined by \( g(x) = \begin{cases} 0 & \text{if } x \in [-1, 0] \\ \sqrt{x} & \text{if } x \in (0, 1] \end{cases} \). Then for a positive integer \( n \), \( \sup_{x \in [-1, 1]} |f_n(x) - g(x)| = \sup_{x \in (0, 1]} |nx - \sqrt{x}| \). To compute the latter, take \( h(x) = nx - \sqrt{x} \), with \( h \) defined on \( [0, \frac{1}{n^2}] \). Then \( h \) is differentiable on \( (0, \frac{1}{n^2}] \), \( h''(x) = n - \frac{1}{2}x^{-3/2} \) for \( x > 0 \), and \( h'(x) = 0 \) when \( x = \frac{1}{4n^2} \). \( h(0) = 0, h(\frac{1}{n^2}) = \frac{1}{n} - \frac{1}{n^2} = 0 \), and

\( h(\frac{1}{3n^2}) = -\frac{1}{4n^2} \), so that \( \sup_{x \in (0, \frac{1}{n^2}]} |nx - \sqrt{x}| = \frac{1}{4n} \), which tends to 0 as \( n \to \infty \), i.e. \( f_n \to g \) uniformly.

We have for \( x \in [-1, 1] \), \( |f_n(x)| \leq n|x| \), so that with \( a_n = n \), the sequence \( \{f_n\} \) satisfies the conditions of the problem, converging to \( g \) uniformly. Suppose, on the contrary, that there exists a constant \( C > 0 \) such that \( |g(x)| \leq C|x| \) for all \( x \in [-1, 1] \). Then \( \sqrt{x} \leq Cx \) for \( x \in (0, 1] \) in particular, or \( \frac{1}{\sqrt{x}} \leq C \) for \( x \in (0, 1] \), an impossibility.

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6. Exercise #9 on p. 166 of Rudin. Let \( \{f_n\} \) be a sequence of continuous functions which converges uniformly to a function \( f \) on a set \( E \). Prove that

\[
\lim_{n \to \infty} f_n(x_n) = f(x)
\]

for every sequence of points \( x_n \in E \) such that \( x_n \to x \), and \( x \in E \). Is the converse of this true?

**Solution.**

Suppose first that \( f_n \) converges to \( f \) uniformly on \( E \). Since \( f_n \) is a sequence of continuous functions, \( f \) is also continuous on \( E \) (uniform limit of a sequence of continuous functions is again continuous). Given any \( \epsilon > 0 \), we can thus find \( N \in \mathbb{Z}^+ \) such that if \( n \geq N \) then \( |f(x) - f(x_n)| < \epsilon/2 \) and also \( \sup_{x \in E} |f(x) - f_n(x)| < \epsilon/2 \) (note that we can arrange for the two inequalities simultaneously by taking the maximum of the two positive integers associated with the inequalities). Then with \( n \geq N \) again, we have \( |f(x) - f_n(x_n)| \leq |f(x) - f(x_n)| + |f(x_n) - f_n(x_n)| < \epsilon/2 + \epsilon/2 = \epsilon \), verifying that \( \lim_{n \to \infty} f_n(x_n) = f(x) \).

The converse is not true. For instance, take a sequence \( \{f_n\}_{n \in \mathbb{Z}^+} \) defined on \( \mathbb{R} \) by

\[
f_1(x) = \begin{cases} \cos(\frac{\pi}{2}x) & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}
\]

and \( f_n(x) = f(x - n) \).

Note that \( f_n(x) = 0 \) if \( x \in (-\infty, -1 + n) \cup (1 + n, \infty) \).

Suppose that \( \lim_{n \to \infty} x_n = x \). Then there exists \( N' \in \mathbb{Z}^+ \) such that if \( n \geq N' \) then \( x_n \in (x - 1, x + 1) \).
Now take any positive integer $N$ such that $-1 + N > \max(x + 1, N')$. Then if $n \geq N, n \geq N'$ in particular and so $x_n < x + 1 < -1 + N \leq -1 + n$. Therefore $f_n(x_n) = 0$. So $\lim_{n \to \infty} f_n(x_n) = 0$, i.e. $f_n \to 0$ point-wise. However, for any $n \in \mathbb{Z}^+$, $\sup_{x \in \mathbb{R}} |f_n(x) - 0| = \sup_{x \in \mathbb{R}} |f_1(x - n)| = \sup_{x \in \mathbb{R}} |f_1(x)| = \sup_{x \in [-1, 1]} |\cos(\frac{\pi}{2} x)| = 1.$