HW #7 SOLUTIONS

1. Let \( f \) and \( g \) be Riemann-integrable complex-valued functions on \([ -\pi, \pi ]\). Define

\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f \overline{g} \, dx \quad \text{and} \quad \| f \| = \sqrt{\langle f, f \rangle} \geq 0.
\]

For integrable functions and \( c \in \mathbb{C} \), prove that:
(a) \( \langle cf_1 + f_2, g \rangle = c \langle f_1, g \rangle + \langle f_2, g \rangle \),
(b) \( \langle f, cg_1 + g_2 \rangle = \overline{c} \langle f, g_1 \rangle + \langle f, g_2 \rangle \),
(c) \( \langle g, f \rangle = \overline{\langle f, g \rangle} \).

(d) If \( f \) is continuous and \( \| f \| = 0 \), then \( f(x) = 0 \) for all \( x \in [-\pi, \pi] \). Remark: We say that \( \langle \cdot, \cdot \rangle \) is a Hermitian inner product on the space of continuous functions.

Solution

(a) \( \langle cf_1 + f_2, g \rangle = \int_{-\pi}^{\pi} (cf_1 + f_2) \overline{g} \, dx = \int_{-\pi}^{\pi} cf_1 \overline{g} + f_2 \overline{g} \, dx = c \int_{-\pi}^{\pi} f_1 \overline{g} \, dx + \int_{-\pi}^{\pi} f_2 \overline{g} \, dx = c \langle f_1, g \rangle + \langle f_2, g \rangle \).
(b) \( \langle f, cg_1 + g_2 \rangle = \int_{-\pi}^{\pi} f (cg_1 + g_2) \overline{dx} = \int_{-\pi}^{\pi} f (cg_1 + g_2) \overline{dx} = \overline{c} \int_{-\pi}^{\pi} f \overline{g_1} \, dx + \int_{-\pi}^{\pi} f \overline{g_2} \, dx = \overline{c} \langle f, g_1 \rangle + \langle f, g_2 \rangle \).
(c) \( \langle g, f \rangle = \int_{-\pi}^{\pi} g \overline{f} \, dx = \int_{-\pi}^{\pi} \overline{g} \, dx = \int_{-\pi}^{\pi} f \overline{g} \, dx = \langle f, g \rangle \).

(d) If \( f : [-\pi, \pi] \rightarrow \mathbb{C} \) is continuous and \( \| f \| = 0 \), then \( \int_{-\pi}^{\pi} |f(x)| \, dx = 0 \). Let \( g : [-\pi, \pi] \rightarrow [0, \infty) \) be defined by \( g(x) = |f(x)| \). It suffices to show that \( g = 0 \).

Suppose for some \( a \in [-\pi, \pi] \), \( g(a) > 0 \). Since \( g \) is continuous, being a composition of two continuous functions \( f \) and the absolute value function, \( g^{-1}(\{g(a)\}, \infty) \) is an open set in \([ -\pi, \pi ]\) containing \( a \). Therefore there exists \( c < d \) such that \( a \in [c, d] \subset [-\pi, \pi] \) and \( g(x) \geq \frac{1}{2} g(a) \) for all \( x \in [c, d] \). We have now

\[
0 = \int_{-\pi}^{\pi} g(x) \, dx \geq \int_{c}^{d} g(x) \, dx \geq \int_{c}^{d} \frac{1}{2} g(a) \, dx = \frac{1}{2} g(a) \cdot (d - c) > 0,
\]

a contradiction.

2. We say that a function \( f \) from \( \mathbb{R} \) to \( \mathbb{C} \) is in \( C^2 \) if it is twice-differentiable, i.e., \( f' \) and \( f'' \) exist, and its second derivative \( f'' \) is continuous. In this case we write \( f \in C^2 \).

(a) Let \( f, g \in C^2 \) be \( 2\pi \)-periodic, i.e., \( f(x + 2\pi e^{i\theta}) = f(x) \) for all \( x \in \mathbb{R} \). Prove that

\[
\langle f'', g \rangle = -\langle f', g' \rangle = \langle f, g'' \rangle,
\]

where the inner product is defined by (1).

(b) Let \( \phi \in C^2 \) be a nonzero \( 2\pi \)-periodic function satisfying \( \phi'' = -\lambda \phi \), where \( \lambda \in \mathbb{C} \). Prove that \( \lambda \in \mathbb{R} \) and \( \lambda \geq 0 \).

(c) Let \( \phi \) be as in part (b). Prove that if \( \phi \) is not constant, then \( \lambda > 0 \).

Solution.

(a) Since \( f(x + 2\pi) = f(x) \) for all \( x \in \mathbb{R} \), differentiating, we have \( f'(x + 2\pi) = f'(x) \) and again \( f''(x + 2\pi) = f''(x) \).
So $f, f', f''$ are all $2\pi$-periodic. By integration by parts, 
\[ \int_{\pi}^{\pi} f''(x) g(x) \, dx = (f'(-\pi)g(-\pi) - f'(-\pi)g'(-\pi)) - \int_{\pi}^{\pi} f'(x) g'(x) \, dx = -\int_{\pi}^{\pi} f'(x) g'(x) \, dx. \]
Note: $g'$ exists and $g' = \overline{g'}$, which follows directly from writing $\overline{g(x)} = g_1(x) + ig_2(x)$ where $g_1 = \text{Re } g$ and $g_2 = \text{Im } g_2$, for $x \in [-\pi, \pi]$.

The other equality follows similarly with integration by parts. (b) By part (a), $-\lambda (\phi, \phi) = (\phi'', \phi) = -\langle \phi', \phi' \rangle$. Since $\phi$ is in $C^2$, it’s in particular continuous, and since it’s nonzero, $\langle \phi, \phi \rangle \neq 0$ by #1d. So $\lambda = \langle \phi', \phi' \rangle / \langle \phi, \phi \rangle$ is real and nonnegative.

(c) Since $\phi'$ is differentiable, it’s continuous, so if $\phi$ is nonconstant then $\phi' \neq 0$, i.e. $\langle \phi', \phi' \rangle \neq 0$ by #1d again. Therefore $\lambda = \langle \phi', \phi' \rangle / \langle \phi, \phi \rangle > 0$.

3. (a) Let $f, g \in C^2$ be $2\pi$-periodic functions satisfying $f'' = -\lambda f$ and $g'' = -\mu g$, where $\lambda, \mu \in \mathbb{R}$. Prove that if $\lambda \neq \mu$, then $(f, g) = 0$.
(b) Recall from ODE theory that the general solution to $\phi'' = -\lambda \phi$, where $\lambda \geq 0$, is
\[ \phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x). \]
Prove that if $\phi$ is $2\pi$-periodic, then $\lambda = n^2$ for some $n \in \mathbb{Z}$.
(c) Let $\phi \in C^2$ be a $2\pi$-periodic function satisfying $\phi' = -\lambda \phi$, where $\lambda \in \mathbb{R}$. Prove that if $\phi$ is not constant, then $\lambda \geq 1$.

Solution.

(a) We have $-\lambda \langle f, g \rangle = \langle f'', g \rangle = \langle f, g'' \rangle = -\mu \langle f, g \rangle$ because $\lambda, \mu$ are reals. If $\lambda \neq \mu$, then $\langle f, g \rangle = 0$ implies that $(f, g) = 0$.

(b) If $c_1$ and $c_2$ are not both 0, then $\phi(x)/\sqrt{c_1^2 + c_2^2}$ can be written as $\sin(\sqrt{\lambda}x + a)$ for some $a \in \mathbb{R}$ by use of the well-known formula $\sin(\theta_1 + \theta_2) = \sin(\theta_1) \cos(\theta_2) + \cos(\theta_1) \sin(\theta_2)$. The function $x \mapsto \sin(\sqrt{\lambda}x + a)$ has a period $\frac{2\pi}{\sqrt{\lambda}}$, so if $\phi$ is $2\pi$-periodic then $2\pi$ must be a positive integer multiple of $\frac{2\pi}{\sqrt{\lambda}}$, i.e. $\frac{2\pi}{\sqrt{\lambda}} n = 2\pi$ for some $n \in \mathbb{Z}$, or $\lambda = n^2$.

(c) Follows directly from part (b) above and from #2(c), for if $n^2 > 0$ then $n^2 \geq 1$ (if $n \in \mathbb{Z}$).

4. Do Exercise #13 on p. 198.

Solution.

Define $f : [0, 2\pi] \to \mathbb{R}$ by
\[ f(x) = \begin{cases} 
  x & \text{if } 0 \leq x < 2\pi \\
  0 & \text{if } x = 2\pi
\end{cases} \]
We apply Parseval’s Theorem, obtaining
\[ \frac{1}{2\pi} \int_0^{2\pi} |f|^2 \, dx = \sum_{n \in \mathbb{Z}} |c_n|^2, \]
where $c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, dx$.
Note that, in general, if $f, g$ are Riemann-integrable functions on a compact interval $[a, b]$ such that $f(x) = g(x)$
for all \( x \in [a, b] \) except at a finite set, then \( \int_a^b f(x) \, dx = \int_a^b g(x) \, dx \). Therefore \( \frac{1}{2\pi} \int_0^{2\pi} |f|^2 \, dx = \frac{1}{2\pi} \int_0^{2\pi} x^2 \, dx = \frac{4\pi^2}{3} \).

c_0 = \frac{1}{2\pi} \int_0^{2\pi} x \, dx = \pi.

For \( n \neq 0 \), \( c_n = \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} \, dx = -\frac{1}{in} \) by integration by parts.

Therefore \( |c_n|^2 = \frac{1}{n^2} \) if \( n \neq 0 \) and \( |c_0|^2 = \pi^2 \).

We have then \( \frac{4\pi^2}{3} = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \), or \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).

5. We say that a function \( f \) from \( \mathbb{R} \) to \( \mathbb{C} \) is in \( C^1 \) if it is continuously differentiable, i.e., \( f' \) exists and \( f' \) is continuous. Recall that if \( g \in \mathcal{R} \),

\[
c_m(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-imx} \, dx.
\]

(a) Let \( f \in C^1 \) be a \( 2\pi \)-periodic function. Prove that

\[
c_m(f') = imc_m(f).
\]

(b) Prove Wirtinger’s inequality: If \( f \in C^1 \) is a \( 2\pi \)-periodic function with \( \int_{-\pi}^{\pi} f(x) \, dx = 0 \), then

\[
\int_{-\pi}^{\pi} |f(x)|^2 \, dx \leq \int_{-\pi}^{\pi} |f'(x)|^2 \, dx,
\]

with equality if and only if \( f(x) = c_1 e^{ix} + c_{-1} e^{-ix} \).

**Solution.**

(a) \( c_m(f') = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-imx} \, dx = \frac{1}{2\pi} \left( f(\pi) e^{-im\pi} - f(-\pi) e^{im\pi} + im \int_{-\pi}^{\pi} f(x) e^{-imx} \, dx \right) = \frac{1}{2\pi} im \int_{-\pi}^{\pi} f(x) e^{-imx} \, dx = imc_m(f) \).

(b) From part (a) and Parseval’s Theorem, the left hand side of the inequality is \( \sum_{n \in \mathbb{Z}} |c_n(f)|^2 \) and the right hand side is \( \sum_{n \in \mathbb{Z}} n^2 |c_n(f)|^2 \), so the inequality is clear. Since \( \int_{-\pi}^{\pi} f(x) \, dx = 0 \), we have \( c_0(f) = 0 \). Therefore the equality holds if and only if \( c_n(f) = 0 \) for all \( |n| \geq 2 \). By Parseval’s Theorem applied to the function \( f - c_1(f) e^{ix} - c_{-1} e^{-ix} \), we have \( \int_{-\pi}^{\pi} |f - c_1(f) e^{ix} - c_{-1} e^{-ix}|^2 \, dx = 0 \) if and only if the equality in Wirtinger’s inequality holds. But \( \int_{-\pi}^{\pi} |f - c_1(f) e^{ix} - c_{-1} e^{-ix}|^2 \, dx = 0 \) if and only if \( f = c_1(f) e^{ix} - c_{-1} e^{-ix} \) by \#1(d) (\( f \) is continuous).