These are supplementary notes for Week 1's lectures. It does not cover all of the material discussed in lectures, but rather is intended to provided extra explanations of material related to the lectures.

The required text is Peter J. Eccles' book *An Introduction to Mathematical Reasoning.* The section and page numbers refer to his book.

I. Mathematical Statements and Proofs

In this part we learn, mostly by example, how to write mathematical statements and how to write basic mathematical proofs.

I.1. The language of mathematics (p. 3)

1.1. Mathematical statements (p. 3)

Definition (p. 3). A **statement** (or **proposition**) is a sentence that is either true or false (both not both).

So '3 is an odd integer' is a statement. But 'π is a cool number' is not a (mathematical) statement. Note that '4 is an odd integer' is also a statement, but it is a false statement.

1.2 Logical connectives (p. 5)

The connective 'or'

Given two statements, such as '2 is an odd integer' (false) and '3 is an odd integer' (true), we can form a combined statement using the connective '**or**' by:

(2 is an odd integer) or (3 is an odd integer).

This statement is true since at least one of the two statements '2 is an odd integer' and '3 is an odd integer' is true.

More abstractly,

Definition (p. 5). Given two statements P and Q, we say that the statement

$$P \text{ or } Q$$

is true if at least one of the two statements P and Q is true.

Table 1.2.1 (p. 6). It is convenient to use a **truth table** to exhibit this definition:

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

For example, the last row says that if P is false and Q is false, then 'P or Q' is false.

The connective 'and'

Definition (p. 7). Given two statements P and Q, we say that the statement

$$P \text{ and } Q$$

is true if both of the two statements P and Q are true. The truth table is (Exercise 1.1 on p. 8):

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

The connective 'not'

Definition (pp. 7–8). Given a statement P, we say that the statement

$$\text{not } P$$

is true if P is false. The truth table is:

P	not P
T	F
F	T

Sets and Venn diagrams

Let A and B be sets (this will be discussed more in section 6). We denote x being an element of A by $x \in A$.

$\cup$ means union and $\cap$ means intersection and ^{c} means complement.

1. $x \in A \cup B$ means $x \in A$ or $x \in B$.
2. $x \in A \cap B$ means $x \in A$ and $x \in B$.
3. $x \in A^c$ means not $x \in A$ (also denoted as $x \notin A$).

So *union* corresponds to *or*, *intersection* corresponds to *and*, and *complement* corresponds to *not*.

I.2. Implications

The following is statement is true for any statements P and Q (proof this as an exercise):

If P, then P or Q.

Let a and b be real numbers. Consider the statement '$ab = 0$'. We claim that this statement is equivalent to the statement '$a = 0$ or $b = 0$'. We can see this as follows.

1. Suppose $ab = 0$.
1a. If $a = 0$, then $a = 0$ or $b = 0$.
1b. If $a \neq 0$, then $0 = \frac{0}{a} = \frac{ab}{a} = b$. Since $b = 0$, we conclude that $a = 0$ or $b = 0$.
We have proved, by cases, that: If $ab = 0$, then $a = 0$ or $b = 0$.
2. Suppose $a = 0$ or $b = 0$.
2a. If $a = 0$, then $ab = 0 \cdot b = 0$.
2b. If $b = 0$, then $ab = a \cdot 0 = 0$.
We have proved, by cases, that: If $a = 0$ or $b = 0$, then $ab = 0$.

Combining what we have done in 1 and 2, we see that the statement '$ab = 0$' is equivalent to the statement '$a = 0$ or $b = 0$'.

One may also see the above by using a truth table:

$a = 0$	$b = 0$	$a = 0$ or $b = 0$	$ab = 0$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

2.1. Implications (p. 10)

Consider the (true) statement:

$$\text{“If an integer } a \text{ is even, then } a^2 \text{ is even.”} \tag{1}$$

This is an example of an **implication**. It means that if the statement 'an integer a is even' is true, then the statement 'a^2 is even' is true.

Remark 1 *To prove (1), we need to prove that* every *even integer a has square a^2 which is even. It is not enough given an example. So, of course, saying* '10 *is even, and* $10^2 = 100$ *is even.' is not enough to prove (1)!*

In this statement, if a is odd (i.e., not even), then nothing is said about a^2. Of course, we can make the separate implication:

$$\text{“If an integer } a \text{ is odd, then } a^2 \text{ is odd.”} \tag{2}$$

This statement is also true, as we shall see below. But it is not equivalent to (does not have the same meaning as) the previous statement.

As a mathematical abbreviation, we write (1) and (2) respectively as:

$$(a \text{ is even}) \;\Rightarrow\; (a^2 \text{ is even})$$

and

$$(a \text{ is odd}) \;\Rightarrow\; (a^2 \text{ is odd}).$$

That is, in lieu of If ..., then ..., we use '$\Rightarrow$'.

More abstractly, given two statements P and Q, when we write:

$$\text{“If } P, \text{ then } Q.\text{”} \tag{3}$$

we mean that if P is true, then Q is true. In particular, if P is false, then nothing is said about whether Q is true. For (3) we also write $P \Rightarrow Q$.

Here's the truth table for implication:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(4)

Let's consider some examples.

Example 1. Consider the predicates $P(n)$ being 'n is even' and $Q(n)$ being 'n^2 is even'. The (universal) statement

$$(n \text{ is even}) \Rightarrow (n^2 \text{ is even})$$

is true. Note that

$P(n)$	n	n^2	$Q(n)$	$P(n) \Rightarrow Q(n)$
T	even	even	T	T
F	odd	odd	F	T

Here we used the 1st and 4th rows of the truth table (4) for $P \Rightarrow Q$ to obtain that the last column has two T's.

Example 2. Consider the predicates $P(x)$ being '$x \geq 0$' and $Q(x)$ being '$x^2 \geq 0$'. The statement

$$(x \geq 0) \Rightarrow (x^2 \geq 0)$$

is true. Note that

$P(x)$			$Q(x)$	$P(x) \Rightarrow Q(x)$
T	$x \geq 0$	$x^2 \geq 0$	T	T
F	$x < 0$	$x^2 > 0$	T	T

Here we used the 1st and 3rd rows of the truth table for $P \Rightarrow Q$ to obtain that the last column has two T's.

Definition. (p. 14) The **converse** of the statement "If P, then Q." is the statement:

"If Q, then P."

In other words, the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$.

So the converse of "If an integer a is even, then a^2 is even." is:

"If a^2 is even, then a is even."

(being understood that a is an integer). In general, as in this case, a statement does not have the same meaning as its converse.

Example. Consider the true statement:

If $x \geq 0$, then $x^2 \geq 0$.

Its converse is:

If $x^2 \geq 0$, then $x \geq 0$.

The converse is false since $-1 < 0$ and $(-1)^2 \geq 0$.

Definition 2 *Let P and Q be statements. The statement: $P \Leftrightarrow Q$ means that $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$.*

In other words, we have both: 'If P is true, then Q is true.' and 'If Q is true, then P is true.'

In this case, we also say that 'P if and only if Q' or 'P is equivalent to Q'.

2.2. Arithmetic (p. 15)

Division algorithm (dividing by 2 case). For each integer a there exists unique integers b and r such that

(1) $a = 2b + r$ and
(2) $r = 0$ or $r = 1$.

2.3. Mathematical truth (p. 17)

I.3. Proofs

3.1. Direct proofs (p. 22)

We now prove that the square of an even integer is even.

Proposition 3 (The square of an even integer is even) *Let a be an integer. If a is even, then a^2 is even.*

Proof. Suppose a is even. By definition, there exists an integer q such that $a = 2q$. Then

$$a^2 = (2q)^2 = 2\left(2q^2\right).$$

Since $2q^2$ is an integer, by this equality and the definition of even, we conclude that a^2 is even. □

Remarks on the proof: We first wrote down the definition of even to get '... $a = 2q$.' Then we used this to rewrite a^2 as equal to $2\left(2q^2\right)$, which is 2 times an integer. One of the most basic techniques of proving results is to write down the definition(s) and work toward the desired conclusion (**direct proof**).

Exercise 4 1. *Prove that if a and b are even integers, then $a + b$ is even.*

2. *Prove that if a and b are even integers, then ab is even. Explain why this is a generalization of Proposition 3.*

SOLUTION. (1) Suppose a and b are even. Then there exist integers q, r such that $a = 2q$ and $b = 2r$. Thus

$$a + b = 2(q + r)$$

is even (since $q + r$ is an integer).

This is a generalization of Proposition 3 since that is the special case where $a = b$.

3.2. Constructing proofs backwards (p. 27)

Consider the statement: For any real numbers a, b and c,

$$a^2 + b^2 + c^2 \geq \frac{1}{3}(a + b + c)^2.$$

We can prove this statement by constructing the proof backwards:

$$\begin{aligned}
a^2 + b^2 + c^2 &\geq \frac{1}{3}(a + b + c)^2 \\
&\Longleftarrow a^2 + b^2 + c^2 \geq \frac{1}{3}\left(a^2 + b^2 + c^2 + 2ab + 2ac + 2bc\right) \\
&\Longleftarrow \frac{2}{3}\left(a^2 + b^2 + c^2\right) \geq \frac{1}{3}(2ab + 2ac + 2bc) \\
&\Longleftarrow \frac{2}{3}\left(a^2 + b^2 + c^2 - ab - ac - bc\right) \geq 0 \\
&\Longleftarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc \geq 0 \\
&\Longleftarrow (a - b)^2 + (a - c)^2 + (b - c)^2 \geq 0.
\end{aligned}$$

Since the last statement (inequality) is true for all real numbers a, b and c, by the string of (5) implications, we conclude that the first statement $a^2 + b^2 + c^2 \geq \frac{1}{3}(a+b+c)^2$ is true for all real numbers a, b and c.

Remark. Some algebra facts you should know (follows just from the distributive law):

$$(a+b)^2 = a^2 + b^2 + 2ab,$$
$$(a-b)^2 = a^2 + b^2 - 2ab.$$