A gradient Ricci soliton (GRS) to be a quadruple \((M^n, g, f, \lambda)\) satisfying
\[
\text{Rc} + \nabla^2 f = \frac{\lambda}{2} g,
\] (1)
where the constant \(\lambda = 1, 0\) and \(-1\) corresponds to being a shrinking, steady and expanding GRS, respectively.

Note that \(\mathcal{L}_{\nabla f} g = 2 \nabla^2 f\).

**Exercise 1** Show that if \((M^n, g, f, \lambda)\) is a GRS and \(\varphi : N^n \to M\) is a diffeomorphism, then \((N, \varphi^*g, f \circ \varphi, \lambda)\) is also a GRS.

**Examples:**
(1) The Gaussian soliton is \((\mathbb{R}^n, g_{\text{euc}}, f, \lambda)\), where \(g_{\text{euc}} = \sum_{i=1}^n dx^i \otimes dx^i\) and \(f(x) = \frac{\lambda}{4} |x|^2\).
(2) An Einstein manifold is \((M^n, g, f, \lambda)\) with \(\text{Rc} = \frac{\lambda}{2} g\) and \(f = 0\). For example, when \(\lambda = 1\), we have the round sphere shrinker \((S^n, g, 0)\), where \(g = 2(n-1) g_{S^n}\), where \(g_{S^n}\) is the standard constant sectional curvature 1 metric.
(3) Product soliton. If \((M_1^n, g_1, f_1, \lambda)\) and \((M_2^n, g_2, f_2, \lambda)\) are GRS, then
\[
(M_1 \times M_2, g_1 + g_2, f_1 \times f_2, \lambda),
\] (2)
where \(g_1 + g_2\) is the product metric, \(f_1 \times f_2 \equiv f_1 \circ p_1 + f_2 \circ p_2\), and \(p_i : M_1 \times M_2 \to M_i, i = 1, 2\), are the projections, is a GRS. Examples which arise in singularity analysis for the Ricci flow are the round cylinder shrinkers \((S^k \times \mathbb{R}^{n-k}, g_{\text{cyl}}, f_{\text{cyl}}, 1), k \geq 2\), where
\[
g_{\text{cyl}}(x, y) = 2(k-1) g_{S^k}(x) + g_{\text{euc}}(y) \quad \text{and} \quad f_{\text{cyl}}(x, y) = \frac{|y|^2}{4}.
\]
(4) Quotient soliton. Suppose that \((M^n, g, f, \lambda)\) is a GRS and \(\Gamma\) is a discrete group of isometries of \(g\) acting freely and properly discontinuously and such that \(f \circ \gamma = f\) for all \(\gamma \in \Gamma\). Let \(g_{\text{quo}}\) denote the quotient Riemannian metric on \(M/\Gamma\) and let \(f_{\text{quo}}\) denote the unique (smooth) function on \(M/\Gamma\) satisfying \(f_{\text{quo}} \circ \pi = f\), where \(\pi : M \to M/\Gamma\) is the quotient map. Then \((M^n/\Gamma, g_{\text{quo}}, f_{\text{quo}}, \lambda)\) is a GRS. For example, the \(\mathbb{Z}_2\)-quotient of \((S^{n-1} \times \mathbb{R}, g_{\text{cyl}}, f_{\text{cyl}})\), where the \(\mathbb{Z}_2\)-action is generated by the involution \((x, r) \mapsto (-x, -r)\), is a shrinking GRS.
(5) Covering soliton. If \((M^n, g, f, \lambda)\) is a GRS and if \(\pi : \tilde{M} \to M\) is a covering space, then \((\tilde{M}, \tilde{g}, \tilde{f}, \lambda)\) is a GRS, where \(\tilde{g} = \pi^* g\) and \(\tilde{f} = f \circ \pi\), called a covering GRS.

**Theorem 2**
1. There is a unique nonflat complete rotationally symmetric steady GRS, called the Bryant soliton.
2. Any complete rotationally symmetric shrinking GRS must be the Gaussian shrinking GRS on \(\mathbb{R}^n\), the round cylinder shrinker on \(S^{n-1} \times \mathbb{R}\), or the round sphere shrinker on \(S^n\).

**Real analyticity:**
Theorem 3 If \((\mathcal{M}^n, g, f, \lambda)\) is a GRS, where \(g\) is \(C^{2,\alpha}\) and \(f\) is \(C^{2,\alpha}\), then both \(g\) and \(X\) are real analytic.

Closed surfaces:

Theorem 4 If \((\mathcal{M}^2, g, f)\) is a GRS on a closed surface, then \(g\) has constant curvature.

Noncompact surfaces:

Theorem 5 If \((\mathcal{M}^2, g, f)\) is a complete nonflat steady GRS, then it is the cigar soliton.

Theorem 6 There does not exist a complete nonflat shrinking GRS on a noncompact surface.

Theorem 7 If \((\mathcal{M}^2, g, f)\) is a complete nonflat expanding GRS, then \(g\) and \(f\) are rotationally symmetric.

Basic equations:

By tracing (1), we obtain
\[
R + \Delta f = \frac{n \lambda}{2}.
\] (3)

Taking the divergence of (1) while applying the contracted second Bianchi identity yields
\[
\frac{1}{2} dR + \Delta (df) = 0.
\]

By the commutator formula
\[
\Delta(du) = d(\Delta u) + \text{Rc}(\nabla u)
\] (4)

for any function \(u\) and by (3), we have
\[
0 = \frac{1}{2} dR + d(\Delta f) + \text{Rc}(\nabla f) = -\frac{1}{2} dR + \text{Rc}(\nabla f).
\]

Thus
\[
2 \text{Rc}(\nabla f) = \nabla R.\] (5)

A useful consequence of this is
\[
\langle \nabla f, \nabla R \rangle = 2 \text{Rc}(\nabla f, \nabla f).\] (6)

Now by (1), for any vector field \(V\),
\[
V(|df|^2) = 2 \langle \nabla_V df, df \rangle
= 2 \left( - \text{Rc}(V) + \frac{\lambda}{2} g(V), df \right)
= (-2 \text{Rc}(\nabla f) + \lambda df)(V),
\]
so that
\[
\nabla |\nabla f|^2 = -2 \text{Rc}(\nabla f) + \lambda \nabla f.\] (7)

Combining this with (5) yields
\[
\nabla (R + |\nabla f|^2 - \lambda f) = 0.\] (8)
Since $\mathcal{M}$ is connected, we conclude that

$$R + |\nabla f|^2 - \lambda f = C,$$  \hspace{1cm} (9)

where $C$ is a constant. If $\lambda = \pm 1$, then by adding a constant to the potential function $f$ we may assume that $C = 0$. If $\lambda = 0$ and $g$ is nonflat, then by scaling the metric we may take $C = 1$.

Define the $f$-Laplacian by

$$\Delta_f \equiv \Delta - \nabla f \cdot \nabla.$$  \hspace{1cm} (10)

(1) For a shrinking GRS, we have

$$R + |\nabla f|^2 = f,$$  \hspace{1cm} (11)

and

$$\Delta_f f = \frac{n}{2} - f.$$  \hspace{1cm} (12)

(2) For a nontrivial steady GRS, we have

$$R + |\nabla f|^2 = 1,$$  \hspace{1cm} (13)

and

$$\Delta_f f = -1.$$  \hspace{1cm} (14)

(3) For an expanding GRS, we have

$$R + |\nabla f|^2 = -f,$$  \hspace{1cm} (15)

and

$$\Delta_f f = f - \frac{n}{2}.$$  \hspace{1cm} (16)

By taking the divergence of $2 \text{Rc}(\nabla f) = \nabla R$, we obtain

$$\Delta_f R = -2|\text{Rc}|^2 + \lambda R.$$  \hspace{1cm} (17)

**Theorem 8** Any steady or expanding GRS on a closed manifold is Einstein.

**Proof.** Since $\mathcal{M}$ is compact and the GRS is steady or expanding, we have $\lambda = \frac{2r}{n} \leq 0$, where $r = \int_{\mathcal{M}} R d\mu / \int_{\mathcal{M}} d\mu$. So, from (17), we have

$$\Delta_f R = -2|\text{Rc}|^2 + \frac{2r}{n} R$$

$$= -2|\text{Rc} - \frac{r}{n} g|^2 - \frac{2r}{n} (R - r).$$  \hspace{1cm} (18)

Then, from $\int_{\mathcal{M}} \Delta_f R e^{-f} d\mu = 0$, we obtain

$$\int_{\mathcal{M}} \left(|\text{Rc} - \frac{r}{n} g|^2 + \frac{r}{n} (R - r)\right) e^{-f} d\mu = 0.$$  

On the other hand, $R - r = -\Delta f$ and $\int_{\mathcal{M}} \Delta f e^{-f} d\mu = \int_{\mathcal{M}} |\nabla f|^2 e^{-f} d\mu$. We have that

$$\int_{\mathcal{M}} \left(|\text{Rc} - \frac{r}{n} g|^2 - \frac{r}{n} |\nabla f|^2\right) e^{-f} d\mu = 0.$$  

Since $r \leq 0$, we conclude that $\text{Rc} = \frac{r}{n} g$. $\blacksquare$