Theorem. For any $\epsilon > 0$, there exists $\delta = \delta(n, \epsilon) > 0$ such that, if $\text{Re} \geq -(n-1)\delta$, $r \in (0, 1]$, and if $u : B_{10r}(p) \to \mathbb{R}^k$ is a $(k, \delta)$-splitting map, then there exists a complete metric length space $(Z, d_Z, z^*)$ such that

$$d_{GH} \left( (B_r(p), d, p), (B_r(z^*, 0^k), d_Z \times \mathbb{R}^k, (z^*, 0^k)) \right) < 4\epsilon r.$$

By rescaling, we assume $r = 1$ throughout.

Lemma 1. Suppose $x, z, w \in B_2(p)$ satisfy $|u(x) - u(z)| \leq \delta$ and $|u(z) - u(w)| - d(z, w) \leq \delta$. Then$$|d^2(x, z) + d^2(z, w) - d^2(x, w)| \leq \Psi(\delta |n|).$$

Remark. Since $|u(z) - u(w)| \leq d(z, w) + \delta$, the the condition $|u(z) - u(w)| - d(z, w) \leq \delta$ is equivalent to

$$|u(z) - u(w)| \geq d(z, w) - \delta.$$That is, the change in $u$ is almost maximal along any geodesic from $z$ to $w$, which is a way of measuring how close $z, w$ are to lying on the same line $\{z\} \times \mathbb{R}$.

Proof. First suppose $k = 1$. Without loss of generality, $d^2(x, w) \geq \Psi(\delta |n|)$. Set

$$h := |\nabla^2 u|^2 + ||\nabla u|^2 - 1|,$$so that the segment inequality and volume comparison give

$$\int_{B_{\frac{1}{2}}(p) \times B_{\frac{1}{2}}(p)} \mathcal{F}_h(x_1, x_2)d(\mu \otimes \mu)(x, y) \leq \int_{B_1(p)} h d\mu \leq C(n)\delta,$$where $\mathcal{F}_h(x_1, x_2) = \int_0^{t_1} d(t)dt$, and $\gamma_{x_1, x_2}$ is the unique minimizing arclength geodesic from $x_1$ to $x_2$ for an open set of full measure in $B_2(p) \times B_2(p)$. Similarly,

$$\int_{B_{\frac{1}{2}}(p) \times B_{\frac{1}{2}}(p) \times B_{\frac{1}{2}}(p)} \mathcal{F}_{\mathcal{F}_h(x_1, \cdot)}(x_2, x_3)d(\mu \otimes \mu \otimes \mu)(x, y, z) \leq C(n) \int_{B_{1}(p) \times B_{1}(p)} \mathcal{F}_h(x_1, y)d(\mu \times \mu)(x_1, y)$$

$$\leq C(n) \int_{B_{10r}(p)} h d\mu \leq C(n)\delta.$$

Claim: There exist $\tilde{x}, \tilde{z}, \tilde{w} \in B_2(p)$ such that the following hold:

1. $d(\tilde{x}, x) + d(\tilde{z}, z) + d(\tilde{w}, w) + ||\nabla u(\tilde{w}) - 1|| \leq \Psi(\delta |n|),$  

2. $\sigma := \gamma_{\tilde{w}, \tilde{z}}$ is unique, and $\mathcal{F}_{\tilde{w}}(\sigma, \tilde{z}) = \int_0^{\tilde{d}} h(\sigma(s))ds \leq \Psi(\delta |n|),$ where $\tilde{d} := d(\tilde{w}, \tilde{z}),$

3. There exists an open subset $\tilde{U} \subseteq [0, \tilde{d}]$ of full measure such that, for all $s \in \tilde{U}$, the minimal geodesic $\tau_s : [0, \ell(s)] \to M$ from $\tilde{x}$ to $\sigma(s)$ is unique,

4. $\mathcal{F}_{\tilde{w}}(\sigma, \cdot)(\tilde{w}, \tilde{z}) = \int_0^{\tilde{d}} \mathcal{F}_{\tilde{w}}(\sigma(s))d\tau(s) d\sigma(\tilde{w}, \tilde{z}) \leq \Psi(\delta |n|).$ 

In fact, by volume comparison and $\int_{B_{10r}(p)} h d\mu \leq \Psi(\delta |n|)$, there is a subset of $\tilde{w} \in B_{\Psi(\delta |n|)}(w)$ of measure at least $c(n)\Psi(\delta |n|)$ such that $||\nabla u(\tilde{w}) - 1|| \leq \Psi(\delta |n|).$ Thus, there is a subset of $\tilde{x} \in B_{\Psi(\delta |n|)}(x)$ of measure at least $c(n)\Psi(\delta |n|)$ such that (2). Then, there is a subset of $\tilde{x} \in B_{\Psi(\delta |n|)}(x)$ of measure at least $c(n)\Psi(\delta |n|)$ such that (3), (4) hold. $\square$
By composing with a translation, we can assume \( u(z) > u(w) = 0 \). Then, by (1) and \( |\nabla u| \leq 1 + \delta \), we have \( |u(\tilde{w})| \leq \Psi(\delta|n|) \) and \( |(u(\tilde{z}) - u(\tilde{w})) - \tilde{d}| \leq \Psi(\delta|n|) \). In particular,
\[
\left| \int_0^\tilde{d} (\nabla u(s), \dot{\sigma}) - (\sigma) - 1 \right| ds \leq \Psi(\delta|n|),
\]
but since \(|\nabla u, \dot{\sigma}| - 1 \leq \delta\), this implies
\[
\int_0^\tilde{d} |\nabla u(s), \dot{\sigma}(s) - 1| ds \leq \Psi(\delta|n|).
\]
In particular,
\[
|u(\sigma(t)) - u(\sigma(0)) - t| = \left| \int_0^t ((\nabla u(s), \dot{\sigma}(s)) - 1) ds \right| \leq \Psi(\delta|n|),
\]
so \(|u(\sigma(t)) - t| \leq \Psi(\delta|n|)\). Also, since \( \int_0^\tilde{d} |\nabla u(t)|^2 - 1 |(\sigma)(s)) ds \leq \Psi(\delta|n|),
\[
\int_0^\tilde{d} |\nabla u(s)| - \dot{\sigma}(s)| ds \leq 10 \left( \int_0^\tilde{d} |\nabla u(s)| - \dot{\sigma}(s)|^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq 10 \left( \int_0^\tilde{d} (|\nabla u(s)|^2 + 1 - 2(\nabla u(s), \dot{\sigma}(s))) ds \right)^{\frac{1}{2}}
\]
\[
\leq 10 \left( \Psi(\delta|n|) + 2 \int_0^\tilde{d} |1 - (\nabla u(s), \dot{\sigma}(s))| ds \right)^{\frac{1}{2}}
\]
\[
\leq \Psi(\delta|n|).
\]
The first variation formula gives \( \ell'(s) = \langle \dot{\sigma}(s), \dot{\tau}_s(\ell(s)) \rangle \), so that
\[
\frac{1}{2} (\ell'(\tilde{d}) - \ell'(0)) = \int_0^\tilde{d} \ell'(s) \ell(s) ds
\]
\[
= \int_0^\tilde{d} \langle \dot{\sigma}(s), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds
\]
\[
= \int_0^\tilde{d} \langle \nabla u(\tau_s(\ell(s))), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds + \int_0^\tilde{d} \langle \dot{\sigma}(s) - \nabla u(\sigma(s)), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds
\]
\[
= \int_0^\tilde{d} \int_0^{\ell(s)} \langle \nabla u(\tau_s(\ell(t))), \dot{\tau}_s(\ell(s)) \rangle dt ds + \int_0^\tilde{d} \langle \dot{\sigma}(s) - \nabla u(\sigma(s)), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds
\]
\[
= \int_0^\tilde{d} \int_0^{\ell(s)} \langle \nabla u(\tau_s(\ell)), \dot{\tau}_s(t) \rangle dt ds + \int_0^\tilde{d} \langle \dot{\sigma}(s) - \nabla u(\sigma(s)), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds
\]
\[
+ \int_0^\tilde{d} \int_0^{\ell(s)} \nabla^2 u(\dot{\tau}_s(r), \dot{\tau}_s(r)) dr dt ds.
\]
However, we can estimate
\[
\left| \int_0^1 \int_0^1 (\nabla u(\tau_s(t), \tau_s(t))) dt ds + \frac{1}{2} d^2 \right| = \left| \int_0^1 (u(\sigma(s)) - u(\bar{x})) ds + \frac{1}{2} d^2 \right| \\
\leq \left| \int_0^1 (s - \bar{d}) ds + \frac{1}{2} d^2 \right| + \Psi(\delta|n) \\
\leq \Psi(\delta|n),
\]
so the claim follows by (1) – (4).

Now suppose \( k \neq 1 \). Suppose \( x, z, w \in B_2(p) \) satisfy \( |u(x) - u(z)| \leq \delta \) and \( ||u(z) - u(w)|| - d(z, w) \leq \Psi(\delta|n) \). Then
\[
u^* := \sum_{j=1}^k \frac{u_j(w) - u_j(z)}{|u(w) - u(z)|} \delta_j,
\]
is a \( C(\delta) \)-splitting function with \( |u^*(w) - u^*(z)| = |u(w) - u(z)| \geq d(w, z) - \Psi(\delta|n) \) and \( |u^*(x) - u^*(z)| \leq \delta \). Thus we can apply the previous lemma to get
\[
|d^2(x, z) + d^2(z, w) - d^2(z, w)| \leq \Psi(\delta|n).
\]

Lemma 2. For any \( \epsilon > 0 \), there exists \( \delta = \delta(\epsilon, \epsilon) > 0 \) such that, for any \( x \in B_2(p) \) and \( v \in \mathbb{R}^k \) with \( |v - u(x)| \leq 1 \), we can find \( y \in B_3(p) \) such that \( ||u(x) - u(y)|| - d(x, y) \leq \epsilon \) and \( |u(y) - v| \leq \epsilon \).

Remark 3. The idea is to find a vector \( w \in TM \) such that \( \langle w, \nabla u_i \rangle \) is close to \( v_i \), and such that the increase of \( u_i \) along \( \gamma_w \) is well-approximated by \( \langle \nabla u_i, w \rangle \).

Proof. Fix \( \mu \in (0, 10^{-2}) \) to be determined. Without loss of generality, \( u(x) = 0 \). Set \( l := |v| \), so that \( \text{(by volume comparison)} \)
\[
\int_{SB_\mu(x)} |\nabla u_i, w| \leq l \int_{B(x, \mu)} |\nabla^2 u_i| d\mu \leq l \Psi(\delta|\mu, n)
\]
for \( 1 \leq i \leq k \). Therefore, for \( \delta = \delta(\mu) > 0 \) sufficiently small, there exist \( x' \in B_\mu(x) \) such that
\[
\int_{S_{x'}M} |\nabla u_i, w| - \frac{u_i(\gamma_w(l)) - u_i(\gamma_w(0))}{l} d\nu(w) \leq l \cdot \Psi(\delta|\mu, n).
\]
Thus there exists \( w \in S_{x'}M \) such that
\[
||v|\langle \nabla u_i(x'), w \rangle - (u_i(\gamma_w(l)) - u_i(\gamma_w(0)))| \leq \Psi(\delta|\mu, n),
\]
\[
|v|\langle \nabla u_i(x'), w \rangle - v_i| < \mu.
\]
Set \( y := \gamma_w(l) \), so that \( |u_i(y) - u_i(x') - v_i| < \mu \), hence \( |u_i(y) - v_i| < (1 + \delta)\mu \) for \( 1 \leq i \leq k \), and \( |u(y) - v| \leq C(\mu) \). On the other hand,
\[
|u(y) - u(x)| \geq |v| - C(\mu) \geq d(x', y) - C(\mu) \geq d(x, y) - C(\mu) \mu.
\]
The claim follows by choosing \( \mu \) small, then \( \delta \) small. □
Proposition 4. For $\delta = \delta(n, \epsilon) > 0$ sufficiently small, there exists a complete metric length space $(Z, d_Z, z^*)$ such that

$$d_{GH}(B_1(p), d, p), (B_1(z^*, 0^k), d_{\mathbb{R}^k}, (z^*, 0))) < \epsilon,$$

(not proven below: in fact there exists $\phi : B_1(p) \to Z$ such that $(\phi, u) : B_1(p) \to Z \times \mathbb{R}^k$ can be taken to be a corresponding $\epsilon$-Gromov-Hausdorff approximation).

Proof. Slight change of notation: $(u_j)$ will be a sequence of $(k, \delta_j)$-splitting maps, and $u_j = (u_j^1, \ldots, u_j^k)$. Suppose by way of contradiction that, for some $\epsilon > 0$, there exist $\delta_j \to 0$ and $(M_j^p, g_j, p_j)$ such that $Rc_{g_j} \geq -(n-1)\delta_j$ and there exist $(k, \delta_j)$-splitting maps $u_j : B_{10}(p_j) \to \mathbb{R}^k$, but that for all $j \in \mathbb{N}$, there does not exist a metric length space $(Z, d_Z, z^*)$ such that

$$d_{GH}(B_1(p_j), B_1(z^*, 0^k)) < \epsilon.$$

Passing to a subsequence, we can assume that $(M_j, g_j, p_j)$ converge in the pointed Gromov-Hausdorff sense to a metric length space $(X_{\infty}, d_{\infty}, x_{\infty})$, via $j^{-1}$-Gromov-Hausdorff approximations $\psi_j : B_j(x_{\infty}) \to B_j(x_j)$. Without loss of generality, $u_j(p_j) = 0$, so because $u_j$ are $(1 + \delta_j + j^{-1})$-Lipschitz, we can also assume that $u_j \circ \psi_j \to u_{\infty}$ locally uniformly for some $1$-Lipschitz function $u_{\infty} : B_{10}(x_{\infty}) \to \mathbb{R}^k$ with $u_{\infty}(x_{\infty}) = 0$.

Fix $x \in u_{\infty}^{-1}(0^k)$ and $v \in B_1(0^k)$, so that $\psi_j(x) \in B_2(p_j)$ for $j \in \mathbb{N}$ sufficiently large. Then there exist $y_j \in B_5(p_j)$ such that $|u_j(y_j) - u_j(\psi_j(x)) - d(y_j, \psi_j(x))| \leq \Psi(\delta_j, n)$ and $|u_j(y_j) - v| \leq \Psi(\delta_j, n)$. Thus there are $z_j \in B_{5+j^{-1}}(x_{\infty})$ with $|u_j(\psi_j(z_j)) - v| \leq \Psi(\delta_j, j^{-1}n)$ and $|u_j(\psi_j(z_j)) - u_j(\psi_j(x)) - d(\psi_j(z_j), \psi_j(x))| \leq \Psi(\delta_j, j^{-1}n)$.

Some subsequence of $(z_j)$ therefore converges to $y \in B_5(x_{\infty})$ with $u_{\infty}(y) = v$ and $u_{\infty}(y) = d_{\infty}(x) = d(y, x)$.

If, for $l = 1, 2$, points $x, y_l \in B_{10}(x_{\infty})$ satisfy $d(y_l, x) = |u_l(x) - u_l(y_l)|$ and $u_{\infty}(y_l) = v$, then

$$|d(\psi_l(y_l), \psi_l(x)) - u_l(\psi_l(x)) - u_l(\psi_l(y_l))| \to 0,$$

$$|u_l(\psi_l(y_l)) - v| + |u_l(\psi_l(x))| \to 0,$$

so Lemma 1 implies that

$$|d^2(\psi_l(y_l), \psi_l(x)) + d^2(\psi_l(y_2), \psi_l(x)) - d^2(\psi_l(y_1), \psi_l(x))| \to 0,$$

hence $d(y_1, y_2) = \lim_{i \to \infty} d(\psi_l(y_1), \psi_l(y_2)) = 0$. That is, $y_1 = y_2$.

We may therefore define $Z := u_{\infty}^{-1}(0^k)$ (with the restricted metric, not a priori a length metric) and

$$\Phi : (Z \cap B_5(x_{\infty})) \times B_1(0^k) \to X_{\infty}$$

by letting $\Phi(x, v)$ be the unique point in $X_{\infty}$ with $d(\Phi(x, v), x) = |u(x) - u(\Phi(x))|$ and $u(\Phi(x, v)) = v$. Similar reasoning as above shows that $\Phi(B_5 \times \mathbb{R}^k(x_{\infty}, 0^k)) \supseteq B_{1}(x_{\infty})$. For any $(x_1, v_1), (x_2, v_2) \in B_5 \times \mathbb{R}^k(x_{\infty}, 0^k)$, with $|v_2| > |v_1| \geq 0$, Lemma 1 gives

$$|d^2(\psi_l(\Phi(x_1, v_1)), \psi_l(\Phi(x_1, 0^k))) + d^2(\psi_l(\Phi(x_1, 0^k), \psi_l(\Phi(x_2, 0^k))) - d^2(\psi_l(\Phi(x_2, 0^k), \psi_l(\Phi(x_1, v_1))))| \to 0,$$
so that
\[ 0 = d^2(\Phi(x_1, v_1), \Phi(x_1, 0^k)) + d^2(\Phi(x_1, 0^k), \Phi(x_2, 0^k)) - d^2(\Phi(x_2, 0^k), \Phi(x_1, v_1)) \]
\[ = |v_1|^2 + d^2_2(x_1, x_2) - d^2(\Phi(x_2, 0^k), \Phi(x_1, v_1)). \]

If we define
\[ u^*_i := \sum_{j=1}^{k} \frac{u^j_i(\psi_i(\Phi(x_2, v_2)))}{|u_i(\psi_i(\Phi(x_2, v_2)))|} u^j_i : B_{10}(x_i) \to \mathbb{R}, \]
\[ u^* := \sum_{j=1}^{k} \frac{u^j_\infty(\Phi(x_2, v_2))}{|u_\infty(\Phi(x_2, v_2))|} u^j_\infty : B_{10}(x_\infty) \to \mathbb{R}, \]
then \( u^*_i \circ \psi_i \to u^* \) in the Gromov-Hausdorff sense, \( u^*_i \) are \((k, C(n)\delta)\)-splitting maps, and
\[ |u^*_\infty(\Phi(x_2, v_2)) - u^*_\infty(x_2)| = |u^\infty(\Phi(x_2, v_2)) - u^\infty(x_2)| = d(\Phi(x_2, v_2), x_2). \]

Also, since \( u^\infty(\Phi(x_2, v_2)) = |v_2| > |v_1| \geq u^\infty(\Phi(x_1, v_1)) \), we can find \( y \in X_\infty \) along a minimal geodesic from \( \Phi(x_2, 0^k) \) to \( \Phi(x_2, v_2) \) such that \( u^\infty(y) = u^\infty(\Phi(x_1, v_1)) \).

Then, since \( u^\infty \) is 1-Lipschitz, we have
\[ d(y, x_2) - |u^\infty(y) - u^\infty(x_2)| = 0 = d(y, \Phi(x_2, v_2)) - |u^\infty(y) - u^\infty(\Phi(x_2, v_2))| \]
\[ \text{hence} \]
\[ |d(\psi_i(\Phi(x_2, v_2)), \psi_i(y)) - u^*_i(\psi_i(\Phi(x_2, v_2))) - u^*_i(\psi_i(y))| \to 0, \]
\[ |d(\psi_i(y), \psi_i(\Phi(x_2, v_2))) - u^*_i(\psi_i(y))| \to 0. \]

We can therefore apply Lemma 1 to obtain
\[ |d^2(\psi_i(\Phi(x_2, v_2)), \psi_i(y) + d^2(\psi_i(y), \psi_i(\Phi(x_1, v_1))) - d^2(\psi_i(\Phi(x_2, v_2)), \psi_i(\Phi(x_1, v_1)))| \to 0, \]
\[ |d^2(\psi_i(\Phi(x_2, 0^k), \psi_i(y)) + d^2(\psi_i(y), \psi_i(\Phi(x_1, v_1))) - d^2(\psi_i(\Phi(x_2, 0^k)), \psi_i(\Phi(x_1, v_1)))| \to 0. \]

Taking the limit as \( i \to \infty \) gives
\[ d^2(\Phi(x_2, v_2), y) + d^2(y, \Phi(x_1, v_1)) = d^2(\Phi(x_2, v_2), \Phi(x_1, v_1)), \]
\[ d^2(x_2, y) + d^2(y, \Phi(x_1, v_1)) = d^2(x_2, \Phi(x_1, v_1)). \]

Finally, since
\[ d(y, \Phi(x_2, v_2)) = |u^*(y) - u^*(\Phi(x_2, v_2))| = |v_2| - \langle v_1, v_2 \rangle |v_2|^{-1}, \]
\[ d(y, x_2) = \langle v_1, v_2 \rangle |v_2|^{-1}, \]
we can combine expressions to get
\[ d^2(\Phi(x_1, v_1), \Phi(x_2, v_2)) = (|v_2| - \langle v_1, v_2 \rangle |v_2|^{-1})^2 + d^2(\Phi(x_1, v_1), y) \]
\[ = (|v_2| - \langle v_1, v_2 \rangle |v_2|^{-1})^2 + d^2(x_2, \Phi(x_1, v_1)) - \langle v_1, v_2 \rangle |v_2|^{-1} \]
\[ = |v_2|^2 - 2\langle v_1, v_2 \rangle + d^2_2(x_1, x_2) - |v_1| |v_2|^2 + d^2_2(x_1, x_2). \]

That is, \( \Phi \) is an isometry. Finally, for any \( y \in B_1(x_\infty) \), we already showed there exists \((x, v) \in B_1^{Z \times \mathbb{R}^k}(x_\infty, 0^k)\) such that \( \Phi(x, v) = y \). Then
\[ d^2_2(x, x_\infty) + |v|^2 = d^2(\Phi(x, v), x_\infty) < 1, \]
so \((x, v) \in B_1^{Z \times \mathbb{R}^k}(x_\infty, 0^k)\), hence \( \Phi \) is onto. \( \square \)