QUANTITATIVE STRATIFICATION I

Basic Facts About Mod-2 Degree

Suppose $M^n$ is a closed, connected $C^\infty$ manifold, $N$ is a connected $C^\infty$ manifold with or without boundary, and $f \in C^\infty(M,N)$. For any regular values $y,z \in N$ of $f$, we have $\# f^{-1}(y) = \# f^{-1}(z) \mod 2$, so we can define the mod-2 degree $\deg_2(f) := \# f^{-1}(y)$ for any regular value $y \in N$ of $f$. We immediately see that $\deg_2(f) = 0$ whenever $f$ is not surjective. It is not difficult to show that $\deg_2(f) = \deg_2(g)$ for any smoothly homotopic $f,g \in C^\infty(M,N)$. If $f \in C^0(M,N)$, we may define $\deg_2(f) := \deg_2(\tilde{f})$, where $\tilde{f}$ is a smooth map homotopic to $f$. If $f_1, f_2$ are two such smooth maps, then they are smoothly homotopic, so $\deg_2(f_1) = \deg_2(f_2)$.

Ruling out Codimension 1 Singularities

Abuse of notation: a $\delta$-Gromov-Hausdorff (GH) map $f : B_r(p_1) \to B_r(p_2)$ does not have to map into $B_r(p_2)$. Rather, $f(B_r(p_1))$ is contained in a $\delta$-neighborhood of $B_r(p_2)$. We recall the following two lemmas from a previous lecture.

Lemma 1. Suppose $(M^n,g,p)$ is a pointed, complete Riemannian manifold with $Rc \geq -(n-1)\delta$, and let $u : B_{10}(p) \to \mathbb{R}^n$ be an $(n,\delta)$-splitting map with $u(0) = 0^n$. Then there exists $\Omega \subseteq B_{1}(p)$ such that:

(i) $|Vol(\Omega) - \omega_n| \leq \Psi(\delta|\iota|)$,

(ii) for each $x \in \Omega$ and $r \in (0,1]$, $u : B_r(x) \to \mathbb{R}^n$ is an $(n,\Psi(\delta|\iota|))$-splitting map.

Lemma 2. Suppose $(M^n,g,p)$ is a pointed, complete Riemannian manifold with $Rc \geq -(n-1)\delta$. If $u : B_{10}(p) \to \mathbb{R}^n$ is a $(\delta,n)$-splitting map, then $|u(x) - u(y)| - d(x,y) \leq \Psi(\delta|\iota|)r$ for all $x,y \in B_1(p)$.

Lemma 3. There exists $\delta = \delta(n) > 0$ such that the following holds. Suppose $(M^n,g,p)$ is a pointed Riemannian manifold with $Rc \geq -\delta(n-1)$ and $d_{GH}(B_{\delta^{-1}}(p),B_{\delta^{-1}}(\overline{p})) \leq \delta^2$, where $\overline{p} = (0^{n-1},0) \in \mathbb{R}^{n-1} \times \{0,\infty\}$. Then there is a continuous map $f : B_{\delta^{-1}}(p) \to B_{\delta^{-1}}(\overline{p})$ such that:

(i) $f(B_{\delta^{-1}}(p) \setminus B_{\delta^{-1}}(p)) \subseteq \partial B_{\delta^{-1}}(p) \cap (\mathbb{R}^{n-1} \times \{0,\infty\})$,

(ii) there exists $q \in B_{\delta^{-1}}(p)$ and a subset $\Omega \subseteq B_1(q)$ with $Vol(\Omega) \geq \frac{1}{2}\omega_n$ and $f^{-1}(f(x)) = \{x\}$ for any $x \in \Omega$.

Proof. Let $\phi : B_{\delta^{-1}}(\overline{p}) \to B_{\delta^{-1}}(p)$ be a $\delta^2$-GH map, and set $q_j := \phi(\delta^{-1}e_j) \in M^n$, and $b_i := d(q_i,p) - d(q_i,\cdot)$. We claim that $b := (b_1,...,b_n)$ is a $\Psi(\delta|\iota|)$-GH map
$B_{\delta^{-\frac{1}{4}}}(p) \to B_{\delta^{-\frac{1}{4}}}(\overline{\Omega})$. In fact, given $y \in B_{\delta^{-\frac{1}{4}}}(\overline{\Omega})$, we have

$$|b_i(\phi(y)) - y_i| = |d(q_i, p) - d(\phi(\delta^{-1}e_i), \phi(y)) - y_i| \leq 2\delta + |\delta^{-1} - |\delta^{-1}e_i - y| - y_i|$$

$$\leq 2\delta + \frac{|\delta^{-1} - y_i - (\delta^{-1} - y_1)^2 + \left(\sum_{j \neq i} y_j^2\right)^\frac{1}{2}}{\delta^{-1}}$$

$$\leq 2\delta + \frac{|y|^2}{\delta^{-1}} \leq 2\delta + \delta^\frac{1}{4},$$

and similarly

$$|b_i(\phi(y)) - b_i(\phi(z))| - |y - z| = |d(\phi(\delta^{-1}e_i), y) - d(\phi(\delta^{-1}e_i), z)| - |y - z|$$

$$\leq \left|\left((\delta^{-1} - y_1)^2 + \left(\sum_{j \neq i} y_j^2\right)^\frac{1}{2} - (\delta^{-1} - z_1)^2 + \left(\sum_{j \neq i} z_j^2\right)^\frac{1}{2}\right)\right| - |y - z|$$

$$\leq 4\delta + \frac{|y|^2 + |z|^2}{\delta^{-1}} \leq 4\delta + 2\delta^\frac{1}{4}.$$
Proof. Suppose by way of contradiction that there exists \( x \in X \) such a tangent cone at \( x \) is \( \mathbb{R}^{n-1} \times C(Y) \) for some metric space \((Y, d_Y)\) not equal to \( \mathbb{R} \). Note that, by a diagonal argument, \( \mathbb{R}^{n-1} \times C(Y) \) is also a Ricci limit space. In fact, there exist \( x_j \in M_j \) and \( \lambda_j \uparrow \infty \) such that \((M_j, \lambda_j g_j, x_j)\) converge in the pointed Gromov-Hausdorff sense to \( \mathbb{R}^{n-1} \times C(Y) \), so \( \mathbb{R}^{n-1} \times C(Y) \) has Hausdorff dimension \( n \). This implies that \( C(Y) \) has Hausdorff dimension 1. Because the metric product \( Y \times (\frac{1}{2}, 1) \) is bi-Lipschitz to \( B_1(cY) \setminus B_{\frac{1}{2}}(cY) \), it follows that \( Y \) has Hausdorff dimension 0.

That is, \( Y \) is discrete, and since the metric on \( \mathbb{R}^{n-1} \times C(Y) \) is a length metric, we must have \( d([1, y], [1, z]) = 2 \) for any distinct \( y, z \in Y \). That is,

\[
d_Y(y, z) = \cos^{-1}\left(\frac{1 + 1 - 4}{2 \cdot 1 \cdot 1}\right) = \cos^{-1}(-1) = \pi.
\]

In particular, \( H^n(B_1(cY)) = \frac{1}{2}\omega_n \# Y \), but because \( \lambda_j \to \infty \), volume convergence implies that \( H^n(B_1(cY)) \leq \omega_n \). This means \( Y \) consists of at most 2 points, but if it consists of exactly 2 points, then \( C(Y) = \mathbb{R} \), contradicting \( x \in S(X) \). Therefore \( Y \) consists of a single point, hence \( \mathbb{R}^{n-1} \times C(Y) = \mathbb{R}^{n-1} \times [0, \infty) \) is the upper half space in \( \mathbb{R}^n \).

We may therefore apply the previous lemma to \((M_j, \lambda_j g_j, x_j)\) for sufficiently large \( j \in \mathbb{N} \), in order to obtain a map (writing \((M, g, p) := (M_j, \lambda_j g_j, p_j)\) for some fixed \( j \)) \( f : B_{\delta^{-\frac{1}{4}}}(p) \to B_{\delta^{-\frac{1}{4}}}(\bar{p}) \) satisfying (i), (ii). We may choose a regular subdomain \( B \subseteq M \) such that

\[
B_{\delta^{-\frac{1}{4}} - \Psi(\delta|n|)}(p) \subseteq B \subseteq B_{\delta^{-\frac{1}{4}}}(p).
\]

Then, since \( f(\partial B) \subseteq \partial B_{\delta^{-\frac{1}{4}}}(\bar{p}) \), we can construct a continuous map \( \bar{f} \) from the double of \( B \) (a closed \( n \)-dimensional \( C^\infty \) manifold) to the double of \( B_{\delta^{-\frac{1}{4}}}(\bar{p}) \) along \( \partial B_{\delta^{-\frac{1}{4}}}(0^+) \cap (\mathbb{R}^{n-1} \times [0, \infty)) \) (diffeomorphic to \( B_1(0^n) \)), satisfying \( \bar{f}^{-1}(\bar{f}(x)) = \{x\} \) for any \( x \in \Omega \). By choosing a regular value in \( \bar{f}(\Omega) \) (which exists since \( f|\Omega \) cannot have uniformly vanishing gradient), we conclude that \( \bar{f} \) has mod-2 degree 1. On the other hand, the image of \( \bar{f} \) is a compact subset of a noncompact manifold, so \( \bar{f} \) cannot be surjective, which implies that \( \bar{f} \) has mod-2 degree 0, a contradiction.

**Appendix 1: Mod-2 Degree**

**Lemma 5.** If \( y \in \text{Int}(N) \) is a regular value of \( f \), then there is a neighborhood \( V \) of \( y \) in \( N \) consisting only of regular values of \( N \), and such that \( V \to \mathbb{Z}, z \mapsto \# f^{-1}(z) \) is constant.

**Proof.** Suppose \( y \in N \) is a regular value of \( f \). Then \( f^{-1}(y) = \{x_1, \ldots, x_k\} \), and there are neighborhoods \( U_i \) of \( x_i \) and \( V_i \) of \( y \) such that \( f|U_i : U_i \to V_i \) are diffeomorphisms. Thus \( V := \bigcap_{i=1}^{k} V_j \setminus \bigcup_{i=1}^{k} U_i \) is a neighborhood of \( y \) in \( N \). If \( z \in V \), then \( f^{-1}(z) \subseteq \bigcup_{i=1}^{k} U_i \), so \( \# f^{-1}(z) \leq k \). In fact, \( f^{-1}(z) = \{(f(U_i))^{-1}(z), \ldots, (f(U_i))^{-1}(z)\} \), so \( z \) is a regular value of \( f \), and \( \# f^{-1}(z) = k \).

**Lemma 6.** (Homotopy Lemma) If \( f, g : M \to N \) are smoothly homotopic, and \( y \in \text{Int}(N) \) is a regular value of both \( f \) and \( g \), then \( \# f^{-1}(y) = \# g^{-1}(y) \mod 2 \).

**Proof.** Let \( V \) be a neighborhood of \( y \) in \( N \) consisting of regular values of \( f \) and \( g \) such that \( z \mapsto \# f^{-1}(z), z \mapsto \# g^{-1}(z) \) are constant on \( V \). Let \( F : M \times I \to N \) be a smooth homotopy from \( f \) to \( g \). By Sard’s theorem, there exists a regular value
$z \in V$ of $F$, so that $F^{-1}(z)$ is a smooth 1-dimensional manifold with boundary $F^{-1}(z) \cap (M \times \partial I)$. Then

$$\#f^{-1}(z) - \#g^{-1}(z) = \#(F^{-1}(z) \cap (M \times \{0\})) - \#(F^{-1}(z) \cap (M \times \{1\}))$$

is even, since $F^{-1}(z)$ is the disjoint union of manifolds with boundary diffeomorphic to closed intervals and circles.

\[\square\]

\textbf{Lemma 7. (Homogeneity Lemma)} For any regular values $y, z \in N$ of $f$, we have $\#f^{-1}(y) = \#f^{-1}(z) \mod 2$.

\textit{Proof.} Let $G : N \times I \to N$ be an isotopy of $N$ with $G(\cdot, 0) = id_N$, $G(y, 1) = z$. Then

$$F : M \times I \to N, (x, t) \mapsto G(f(x), t)$$

is a smooth homotopy from $f$ to $g := F(\cdot, 1)$, and $g^{-1}(z) = (F \circ f)^{-1}(z) = f^{-1}(y)$, so

$$\#f^{-1}(z) = \#g^{-1}(z) = \#f^{-1}(y) \mod 2.$$