**ALMOST-SPLITTING THEOREM**

**Motivation and Overview**

One of the motivating ideas of Cheeger-Colding theory is to improve rigidity theorems to “almost-rigidity theorems”. By a rigidity theorem, we mean a statement like the following: (Cheeger-Colding)

“if a certain geometric quantity is as large as possible relative to the lower bound on Ricci curvature, then the manifold in question is a warped product metric of a particular type”.

Three examples of rigidity theorems are the following:

- A space with $Rc \geq n - 1$ and $diam(M) = \pi$ must be the sphere (a metric suspension).
- A space with $Rc \geq 0$ containing a line (diameter of asymptotic cone) must split a factor of $\mathbb{R}$.
- A space with $Rc \geq 0$ and $\text{Vol}(B_r(x))/r^n = \text{Vol}(B_s(x))/s^n$ for some $r \geq s$ must be Euclidean on $B_r(x)$.

We focus on the second of these theorems.

**Theorem 1.** (Cheeger-Gromoll) If $(M^n, g)$ is a complete Riemannian manifold with $Rc \geq 0$, and if $\gamma: \mathbb{R} \rightarrow M$ is a geodesic line, then $(M, g)$ is isometric to $(N \times \mathbb{R}, g_N + dt^2)$.

**Proof.** Set $b^\pm(x) := \lim_{t \rightarrow \pm\infty} (d(\gamma(t), x) - t)$, 1-Lipschitz functions. Then $b^+ + b^- \geq 0$ everywhere, $b^+ + b^- = 0$ on $\gamma$, and by Laplacian comparison, $\Delta b^\pm \leq 0$, so the strong maximum principle gives $b^+ + b^- = 0$, hence $\Delta b^\pm = 0$. For any $x \in M$, if $\tilde{c}^\pm$ is an asymptotes at $x$ corresponding to $b^\pm$, then $b^\pm(\tilde{c}^\pm(t)) = -t$, so $|\nabla b^\pm| \equiv 1$, and by Bochner’s formula,

$$0 = \Delta |b^\pm|^2 = 2Rc(\nabla b^+, \nabla b^+) + 2|\nabla^2 b^+|^2.$$ 

Thus $\nabla^2 b^+ \equiv 0$, and if $(\varphi_t)$ is the flow of $\nabla b^+$, then $\{b^+ = 0\} \times \mathbb{R} \rightarrow M, (x,t) \mapsto \varphi_t(x)$ is an isometry. Note that, conversely, if $M = N \times \mathbb{R}$ is a product Riemannian manifold, and $b: M \rightarrow \mathbb{R}$ is the projection, then $|\nabla b| = 1$ and $\nabla^2 b = 0$. □

The corresponding almost-rigidity theorem is the following:

**Theorem.** (Cheeger-Colding) For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n) > 0$ such that the following holds. Let $(M^n, g, p)$ be a pointed Riemannian manifold with $Rc \geq -(n - 1)\delta$, and suppose $\gamma: (-\delta^{-1}, \delta^{-1}) \rightarrow M$ is a minimal geodesic with $\gamma(0) = p$. Then there is a pointed metric length space $(X, d, x)$ such that

$$d_{GH}(B_1(p), B_1(x, 0^\epsilon)) \leq \epsilon.$$ 

**Remark 2.** Proving the almost-rigidity theorem is made difficult for several reasons, especially that $x \mapsto d(x, \gamma(t)) - t$ is not harmonic, and so not smooth. This suggests replacing $u$ with a harmonic approximation, and we will see that this approximation almost satisfies the properties of $b^+$ in an integral sense. Using Colding’s segment inequality, we can integrate along most curves to get a kind of almost-Pythagorean theorem.
Moreover, many of the techniques developed in the proof will be applied in future lectures, and some of the propositions we prove stronger than needed for just proving the almost-splitting theorem. In this lecture and the next, we will show that regions of \((M, g)\) which are close to some metric product \(Z \times \mathbb{R}^k\) can be characterized by the existence of \(k\) harmonic functions which are (in some integral sense) almost-linear and almost-pairwise orthogonal. This correspondence is essential to Colding’s volume convergence theorem, Cheeger-Naber’s resolution of the codimension 4 conjecture, and any detailed analysis of the singular set of Ricci limit spaces.

**Almost-Metric Product Implies \(\epsilon\)-Splitting Maps**

Let \((M^n, g, p)\) be a pointed Riemannian manifold. In agreement with the literature on Ricci limit spaces, we denote by \(\Psi(\epsilon_1, \ldots, \epsilon_k | a_1, \ldots, a_k)\) any quantity that vanishes as \(\epsilon_1, \ldots, \epsilon_k \to 0\), fixing the parameters \(a_1, \ldots, a_k\). We denote by \(d\mu\) the Riemannian volume measure.

**Definition.** A \((k, \epsilon)\)-splitting map is a smooth map \(u = (u_1, \ldots, u_k) : B_r(p) \to \mathbb{R}^k\) satisfying the following:

\((i)\) \(\Delta_g u = 0\),
\((ii)\) \(|\nabla u|_g \leq 1 + \epsilon\),
\((iii)\) \(\int_{B_r(p)} |\langle \nabla u_i, \nabla u_j \rangle - \delta_{ij}| d\mu \leq \epsilon\),
\((iv)\) \(r^2 \int_{B_r(p)} |\nabla^2 u_i|^2 d\mu \leq \epsilon\).

where \(|\nabla u|_g(x)\) is the operator norm of \(du_x : (T_x M, g|_x) \to (T_{u(x)} \mathbb{R}^k, g_{Euc}|_{u(x)})\).

Sometimes, in this definition, \((iii)\) is replaced by the a priori more restrictive assumption

\[\int_{B_r(p)} |\langle \nabla u_i, \nabla u_j \rangle - \delta_{ij}|^2 d\mu \leq \epsilon,\]

which is equivalent due to condition \((ii)\).

**Remark.** It was shown by deTurck and Kazdan that the component functions of a Riemannian metric have optimal regularity in harmonic coordinates, and Anderson showed that, given an injectivity radius lower bound and Ricci bound, it is possible to find harmonic coordinates of fixed size, satisfying uniform estimates. However, it is not possible to find harmonic coordinates given just a Ricci lower bound (or even a 2-sided Ricci bound) and volume bounded below, so one needs weaker hypotheses.

In fact, it is often better to use time-slices of solutions to the heat equation rather than harmonic approximations.

**Remark.** If \((M, g, p) = (N \times \mathbb{R}^k, g + g_{Euc}, (q, 0^k))\), and \(u : B_r(q, 0^k) \to \mathbb{R}^k\) is the restriction of the projection map, then \(u\) is a \((k, 0)\)-splitting map. Conversely, if \(u\) is a \((k, 0)\)-splitting map, then \(\langle \nabla u', \nabla u' \rangle = \delta_{ij}\) and \(\nabla^2 u' = 0\). Assume without loss of generality that \(0^k = \text{im}(u)\), and set \(N := u^{-1}(0^k)\). Let \(\Phi_t^k\) be the flow generated by \(\nabla u'\), and note that \(\nabla u', \nabla u' = \nabla^2 u' - \nabla^2 u' = 0\), so \(\Phi_t^k\) pairwise commute. In particular, the map \(N \times \mathbb{R}^k \to M, (x, t) \mapsto \Phi_{t_x}^k \circ \cdots \circ \Phi_{t_z}^k(x)\) is a Riemannian isometry.

**Theorem.** For any \(\epsilon > 0\), there exists \(\delta = \delta(n, \epsilon) > 0\) such that, if \(Rc(g) \geq -\delta(n - 1)g\) and if \(r \in (0, 1)\) satisfies

\[d_{GH}(B_{S^{-1}r}(p), d_{g}, p), (B_{S^{-1}r}(z, 0^k), \mathbb{R}^k), \mathbb{R}^k) < \delta r\]


for some pointed metric space \((Z, d_Z, z)\), then there exists an \((k, \varepsilon)\)-splitting map \(u : B_r(p) \to \mathbb{R}^k\).

**Proof.** Without loss of generality, \(r = 1\), and \(\delta < 10^{-3}\). Choose \(q_i^+ \in M\) corresponding to \((z^*, \pm \delta^{-1} e_i)\), and set \(b_i^+(x) := d(x, q_i^+)\) and \(b_i^-(x) := d(p, q_i^-) - d(x, q_i^-)\). Choose \(q_i^+ \in M\) corresponding to \(\frac{\delta^{-1}}{\sqrt{t}}(e_i + e_j)\), and set \(b_{ij}^+(x) := d(x, q_i^+) - d(p, q_j^+)\).

**Claim 1:** \(\sup_{B_{10}(p)} \left| \frac{b_i^+(x) + b_j^+(x)}{\sqrt{2}} - b_{ij}^+(x) \right| \leq \Psi(\delta|n|)\) for \(1 \leq i \neq j \leq k\), and \(\sup_{B_{10}(p)} |b_i^+(x) - b_i^-(x)| \leq \Psi(\delta|n|)\) for \(1 \leq i \leq k\).

We first observe that, for any \(v \in S^{k-1}\), we have

\[
\lim_{t \to \infty} (tv - x) = - (x, v),
\]

uniformly on compact subsets of \(\mathbb{R}^k\). Intuitively, this is because the level set of \(x \mapsto |tv - x| - t\) is a sphere tangent to at \(x\), with curvature approaching \(0\) as \(t \to \infty\). Rigorously, we have

\[
|tv - x| - t = \sqrt{(tv - x, v)^2 + (x - (x, v)v)^2} - (tv - x, v) = \sqrt{|x - (x, v)v|^2} - (x, v),
\]

but the first term is bounded in absolute value by

\[
\frac{|x|^2}{2(tv - x, v)} \leq \frac{R^2}{2t - 2R}
\]

for \(x \in B_R(0^k)\). A slight modification of the above computation shows that

\[
\lim_{t \to \infty} (d_{Z \times \mathbb{R}^k}((z, x), (z^*, tv)) - t) = -(x, v),
\]

uniformly for \((z, x) \in B_{20}(z^*, 0^k)\).

Set \(\tilde{b}_i^+(z, x) := |x - \delta^{-1} e_i| - \delta^{-1}\), \(\tilde{b}_i^-(z, x) := \delta^{-1} - |x + \delta^{-1} e_i|\), \(\tilde{b}_{ij}^+(z, x) := |x - \frac{\delta^{-1}}{\sqrt{t}}(e_i + e_j)| - \delta^{-1}\), so that \(\tilde{b}_i^\pm\) is \(\delta\)-close in the Gromov-Hausdorff sense to \(b_i^\pm\), while \(\tilde{b}_{ij}^+\) is \(\delta\)-close to \(b_{ij}^+\). Also, as \(\delta \to 0^+\), we have

\[
\frac{\tilde{b}_i^+(x) + \tilde{b}_j^+(x)}{\sqrt{2}} - \tilde{b}_{ij}^+(x) \to -\langle x, e_i \rangle - \langle x, e_j \rangle + \sqrt{\frac{e_i + e_j}{2}} = 0,
\]

\[
\tilde{b}_i^+(x) - \tilde{b}_i^-(x) \to -\langle x, e_i \rangle - \langle x, -e_i \rangle = 0,
\]

both uniformly in \(B_{20}(z^*, 0^k)\). \(\Box\)

Now let \(u' \in C^\infty(B_{10}(p)) \cap C^0(\overline{B}_{10}(p))\) satisfy

\[
\begin{cases}
\Delta u_t = 0 & \text{in } B_{10}(p) \\
u_i = b_i^+ & \text{on } \partial B_{10}(p)
\end{cases}
\]

**Claim 2:** \(\sup_{B_{10}(p)} |b_i^+ - u_i| \leq \Psi(\delta|n|)\).

By Laplacian comparison, in \(B_{10}(p)\), we have (in the sense of distributions)

\[
\Delta b_i^+ \leq (n - 1)\sqrt{\delta} \coth\left(\frac{1}{2\sqrt{\delta}}\right) d\mu \leq 2(n - 1)\sqrt{\delta} d\mu
\]
Thus $\Delta(u_i - b_i^+) \geq -2(n-1)\sqrt{\delta}dg$ in $B_{10}(p)$, and since $u_i - b_i^+ = 0$ on $\partial B_{10}(p)$, the generalized maximum principle gives

$$\sup_{B_{8}(p)} (u_i - b_i^+) \leq C(n)\sqrt{\delta}$$

Similarly,

$$\Delta(b_i^+ - u_i) \geq -2(n-1)\sqrt{\delta}d\mu,$$

and by Claim 1, we have $|b_i^+ - u_i| = |b_i^- - b_i^+| \leq \Psi(\delta|n|)$ on $\partial B_{10}(p)$, so the generalized maximum principle and Claim 1 give

$$\sup_{B_{8}(p)} (b_i^+ - u_i) \leq \sup_{B_{10}(p)} (b_i^- - u_i) + \Psi(\delta|n|) \leq \Psi(\delta|n|).$$

\[\square\]

**Claim 3:** $f_{B_8(p)} (|\nabla u_i|^2 - 1) dg \leq \Psi(\delta|n|)$.

Let $\varphi \in C_c^\infty(B_0(p))$ be a cutoff function with $\varphi|_{B_8(p)} \equiv 1$ and $|\nabla \varphi| \leq C(n)$. Then, since $\Delta(u_i - b_i^+) \geq -2(n-1)\sqrt{\delta}d\mu$, we can integrate by parts to obtain

$$\int_{B_{10}(p)} (\nabla(u_i - b_i^+))^2 \varphi^2 d\mu \leq C(n) \int_{B_{8}(p)} (\nabla(u_i - b_i^+), \varphi \nabla(u_i - b_i^+)) d\mu$$

$$= -\int_{B_{8}(p)} \varphi^2 (u_i - b_i^+ + \Psi(\delta|n|))(u_i - b_i^+) + C(n) \int_{B_{8}(p)} \varphi |\nabla \varphi| \cdot |u_i - b_i^+| \cdot |\nabla(u_i - b_i^+)| d\mu$$

$$- \Psi(\delta|n|) \int_{B_{8}(p)} 2\varphi (\nabla(u_i - b_i^+), \nabla \varphi) d\mu$$

$$\leq 2(n-1)(\sqrt{\delta} + \Psi(\delta|n|)) \int_{B_{8}(p)} \varphi^2 d\mu + \frac{1}{2} \int_{B_{8}(p)} (\nabla(u_i - b_i^+))^2 \varphi^2 d\mu$$

$$+ C(n) \int_{B_{8}(p)} |u_i - b_i^+|^2 d\mu + \Psi(\delta|n|) \int_{B_{8}(p)} \varphi^2 |\nabla(u_i - b_i^+)|^2 d\mu$$

$$\leq \Psi(\delta|n|) + \left(\frac{1}{2} + \Psi(\delta|n|)\right) \int_{B_{8}(p)} (\nabla(u_i - b_i^+))^2 \varphi^2 d\mu,$$

where we used Cauchy’s inequality in the last step, and the volume doubling property several times. On the other hand, the Cheng-Yau gradient estimate implies $\sup_{B_{8}(p)} |\nabla u_i| \leq C(n)$, so

$$|\nabla u_i|^2 - 1 = |(\nabla u_i - \nabla b_i^+, \nabla u_i + \nabla b_i^-)|$$

$$\leq (1 + C(n))|\nabla(u_i - b_i^+)|,$$

and the claim follows by Hölder’s inequality. \[\square\]

**Claim 4:** $\int_{B_8(p)} (|\nabla u_i, \nabla u_j|) d\mu \leq \Psi(\delta|n|)$.

The same argument as in Claim 3 gives

$$\int_{B_8(p)} \frac{|(\nabla(u_i + u_j)|^2}{2} - 1 d\mu \leq \Psi(\delta|n|).$$

On the other hand, Claim 3 implies

$$\int_{B_8(p)} \frac{|(\nabla(u_i + u_j)|^2}{2} - 1 d\mu = \int_{B_8(p)} \frac{1}{2} |(\nabla u_i|^2 - 1 + \frac{1}{2}|(\nabla u_j|^2 - 1 + (\nabla u_i, \nabla u_j)| d\mu$$

$$\geq \int_{B_8(p)} |(\nabla u_i, \nabla u_j)| d\mu - \Psi(\delta|n|),$$
so the claim follows by combining these inequalities. □

Now fix a cutoff function \( \varphi \in C^\infty_c(B_6(p)) \) with \( \varphi|B_6(p) \equiv 1 \) and \( |\nabla \varphi| + |\Delta \varphi| \leq C(n) \). Then integrating Bochner’s formula

\[
\Delta(|\nabla u_i|^2 - 1) = 2|\nabla^2 u_i|^2 + 2Rc(\nabla u_i, \nabla u_i)
\]

against the cutoff function gives

\[
\int_{B_6(p)} |\nabla^2 u_i|^2 dg \leq C(n) \int_{B_6(p)} |\nabla^2 u_i|^2 d\mu \leq C(n) \int_{B_6(p)} (\delta|\nabla u_i|^2 + |\nabla u_i|^2 - 1) \cdot |\Delta \varphi| d\mu \leq \Psi(\delta|n|).
\]

Now fix \( w \in \mathbb{S}^{k-1} \), and set \( \tilde{u} := \langle u, w \rangle_{\mathbb{R}^k} = \sum_{j=1}^k w_j u_j \), so that \( \tilde{u} \) is harmonic, hence Bochner’s formula implies

\[
\Delta(|\nabla \tilde{u}|^2 - 1) \geq -C(n)\delta.
\]

on \( B_6(p) \). In addition, Claims 3,4 imply

\[
\int_{B_6(p)} |\langle \nabla \tilde{u} \rangle^2 - 1| d\mu = \int_{B_6(p)} \left| \sum_{i,j=1}^k w_i w_j \langle \nabla u_i, \nabla u_j \rangle - 1 \right| d\mu \leq \sum_{i,j=1}^k |w_i w_j| \int_{B_6(p)} |\langle \nabla u_i, \nabla u_j \rangle - \delta_{ij}| d\mu \leq \Psi(\delta|n|),
\]

so we can apply the generalized maximum principle to get

\[
\sup_{B_6(p)} (|\nabla \tilde{u}|^2 - 1) \leq C(n) \int_{B_6(p)} |\langle \nabla \tilde{u} \rangle^2 - 1| d\mu + C(n)\delta \leq \Psi(\delta|n|).
\]

We may therefore conclude that

\[
|\nabla u|_x(x) = \sup_{v \in T_xM, |v| = 1} \sup_{w \in \mathbb{R}^k, |w| = 1} \langle du_x(v), w \rangle
= \sup_{v \in T_xM, |v| = 1} \sup_{w \in \mathbb{R}^k, |w| = 1} \langle \nabla (u, w)_{\mathbb{R}^k}(x), v \rangle
= \sup_{w \in \mathbb{R}^k, |w| = 1} |\nabla (u, w)_{\mathbb{R}^k}(x)| \leq 1 + \Psi(\delta|n|).
\]

\[
\Box
\]

THE EXCESS, AND THE ALMOST-SPLITTING THEOREM

Suppose \( Rc_g \geq -(n-1)\delta \), fix \( p \in M \), set \( r(x) := d(x, p) \), and define the approximate Green’s function \( G(x) := \frac{1}{n\omega_n} \int_r^{\infty} s^{1-n}\delta(s) ds \), as well as the auxiliary function \( U(x) := \int_0^{r^2(x)} s^{1-n}\delta(s) \left( \int_0^r s^{1-n}\delta(t) dt \right) ds \). Then \( \lim_{x \to p} G(x) = \infty \)

\( \langle \nabla G, \nabla r \rangle \leq 0, |\nabla G(x)| = \frac{1}{1-(n-2)\delta r(x)}, and \Delta G \geq 0 \) on \( M \setminus \{ p \} \) in the support sense by Laplacian comparison. Also, \( U(0) = 0, \langle \nabla U, \nabla r \rangle = 0, \ |\nabla U(x)| = \frac{r^2(x)}{a_{n-2}(r)}, and \Delta U \leq 1 \) on \( M \) in the support sense. In the case \( \delta = 0 \), these functions simplify to \( G(x) = C_n r^{2-n}(x) \) and \( U(x) = r^2(x)/2n \). For \( R > 0 \), define \( G_R := G - G(R), U := U - U(R), c_1 := -U'(R)/G'(R), and L_R := c_1 G_R + U_R \), so that \( L_R(R) = L'_{R}(R) = 0, L_R \leq 0 \) on \( B_R(p) \setminus \{ p \} \), and \( \Delta L_R \geq 1 \) on \( B_R(p) \setminus \{ p \} \) in the support.
sense. $L_R$ is a useful lower barrier on annuli with outer radius $R$ because it vanishes there, and is uniformly subharmonic in the interior.

**Theorem 3.** For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n) > 0$ such that the following holds. Let $(M^n, g, p)$ be a pointed Riemannian manifold with $Rc \geq -(n-1)\delta$, and suppose $\gamma : (-\delta^{-1}, \delta^{-1}) \to M$ is a minimal geodesic with $\gamma(0) = p$. Then there is a pointed metric length space $(X, d, x)$ such that

$$\text{d}_{GH}((B_1(p), B_1(x, 0^1))) \leq \epsilon.$$ 

**Proof.** Fix $\delta \in (0, 10^{-2})$ to be determined. Then, by Laplace comparison,

$$\Delta E(x) \leq 2(n-1)\sqrt{\delta} \coth(\sqrt{\delta}(\delta^{-1} - 20)) \leq 2(n-1)\sqrt{\delta} \coth(\frac{1}{2\sqrt{\delta}}) \leq 4(n-1)\sqrt{\delta}$$

on $B_{20}(p)$. Fix $\eta \in (0, 10)$. If $d(x, p) \leq \eta$, then $E(x) \leq 2\eta$. If instead $\eta < d(x, p) < 10$, then $\Delta(E - 4(n-1)\sqrt{\delta}L_{20}) \leq 0$ on $A_{\eta, 20}(x)$, and $E - 4(n-1)\sqrt{\delta}L_{20} \geq 0$ on $\partial B_{20}(x)$. If $\min_{\partial B_{\eta}(x)} E \geq 4(n-1)\sqrt{\delta}L_{20}(\eta)$, then the maximum principle gives $E \geq 4(n-1)\sqrt{\delta}L_{20}(\eta)$, and in particular, $E(p) > 0$, a contradiction. Thus $\min_{\partial B_{\eta}(x)} E \geq 4(n-1)\sqrt{\delta}L_{20}(\eta)$, and so

$$E(x) \leq 2\eta + 4(n-1)\sqrt{\delta}L_{20}(\eta).$$

Now choose $\eta > 0$ to be the unique solution (for $\delta$ sufficiently small) of $4(n-1)\sqrt{\delta}L_{20}(\eta) = 2\eta$, so that $\eta = \Psi(\delta|n)$.

We therefore have $E \leq \Psi(\delta|n)$ on $B_{10}(p)$, so there is a $\Psi(\delta|n)$-splitting map on $B_{10\epsilon(n, \delta)}(p)$, and $B_{\epsilon(n, \delta)}(p)$ is Gromov-Hausdorff close to splitting a factor of $\mathbb{R}$. \(\square\)