THE EXCESS, AND THE ALMOST-SPLITTING THEOREM

Suppose $Rc_g \geq -(n-1)\delta$, fix $p \in M$, set $r(x) := d(x,p)$, and define the approximate Green’s function $G(x) := \frac{1}{2\omega_n} \int_0^\infty \gamma_0^{1-n}(s)ds$, as well as the auxiliary function $U(x) := \int_0^{r(x)} \gamma_0^{1-n}(s)(\int_0^s \gamma_0^{1-n}(t)dt)ds$. Then $\lim_{x \to p} G(x) = \infty$ ($\nabla G, \nabla r \leq 0, |\nabla G(x)| = \frac{1}{(n-2)\omega_{n-1}(r)}$, and $\Delta G \geq 0$ on $M \setminus \{p\}$ in the support sense by Laplacian comparison. Also, $U(0) = 0$, ($\nabla U, \nabla r \geq 0, |\nabla U(x)| = \frac{\omega_{n-1}(r)}{\Delta(r)}$), and $\Delta U \leq 1$ on $M$ in the support sense. In the case $\delta = 0$, these functions simplify to $G(x) = C_n r^{2-n}(x)$ and $U(x) = r^2(x)/2n$. For $R > 0$, define $G_R := G - G(R)$, $U := U - U(R)$, $c_1 := -U'(R)/G'(R)$, and $L_R := c_1G_R + U_R$ so that $L_R(R) = L_R(0) = 0$, $L'R \leq 0$ on $B_R(p) \setminus \{p\}$, and $\Delta L_R \geq 1$ on $B_R(p) \setminus \{p\}$ in the support sense. $L_R$ is a useful lower barrier on annuli with outer radius $R$ because it vanishes there, and is uniformly subharmonic in the interior.

**Theorem 1.** (Cheeger-Colding, Abresch-Gromoll) For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n) > 0$ such that the following holds. Let $(M^n, g, p)$ be a pointed Riemannian manifold with $Rc \geq -(n-1)\delta$, and suppose $\gamma : [-\delta^{-1}, \delta^{-1}] \to M$ is a minimal geodesic with $\gamma(0) = p$. Then there is a $(1, \epsilon)$-splitting map $u : B_{10}(p) \to \mathbb{R}$.

**Proof.** Define the Excess function $E(x) := d(\gamma(-\delta^{-1}), x) + d(x, \gamma(\delta^{-1})) - 2\delta^{-2}$. Fix $\delta \in (0,10^{-2})$ to be determined. Then, by Laplace comparison, $\Delta E(x) \leq 2(n-1)\sqrt{\delta} \coth(\sqrt{\delta}(\delta^{-1} - 20)) \leq 2(n-1)\sqrt{\delta} \coth(\frac{1}{2\sqrt{\delta}}) \leq 4(n-1)\sqrt{\delta}$ on $B_{20}(p)$. Fix $\eta \in (0,10)$ to be determined. If $d(x,p) \leq \eta$, then $E(x) \leq 2\eta$. If instead $\eta < d(x,p) < 10$, then $\Delta(E - 4(n-1)\sqrt{\delta}L_{20}) \leq 0$ on $A_{\eta,20}(x)$, and $E - 4(n-1)\sqrt{\delta}L_{20} \geq 0$ on $\partial B_{20}(x)$. If $\min_{\partial B_{\eta}(x)} E \geq 4(n-1)\sqrt{\delta}L_{20}(\eta)$, then the maximum principle gives $E \geq 4(n-1)\sqrt{\delta}L_{20}(\eta)$, and in particular, $E(p) > 0$, a contradiction. Thus $\min_{\partial B_{\eta}(x)} E \geq 4(n-1)\sqrt{\delta}L_{20}(\eta)$, and so $E(x) \leq 2\eta + 4(n-1)\sqrt{\delta}L_{20}(\eta)$. Now choose $\eta > 0$ to be the unique solution (for $\delta$ sufficiently small) of $4(n-1)\sqrt{\delta}L_{20}(\eta) = 2\eta$, so that $\eta = \Psi(\delta n)$. We therefore have $E \leq \Psi(\delta n)$ on $B_{20}(p)$, so there is a $\Psi(\delta n)$-splitting map on $B_{10}(p)$.

EXISTENCE OF GOOD CUTOFF FUNCTIONS

In the Euclidean setting, if $u \in C^\infty(B_1) \cap C^0(\overline{B_1})$ satisfies $\Delta u \geq \delta$ in $B_1$ and $u = 0$ on $\partial B_1$, then we can apply the maximum principle to $x \mapsto f(x) = \frac{\delta}{2}(|x|^2 - 1)$ to get $f(x) \leq -\frac{\delta}{2}(1 - |x|^2)$ for all $x \in B_1$. That is, $u$ decreases a quantifiable amount as $x$ moves into the interior of $B_1$. The generalization of this technique to manifolds with lower Ricci bounds is makes use of the function $U$. 

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Proposition 2. (Quantitative Maximum Principle) Suppose \((M, g, p)\) is a complete pointed Riemannian manifold with \(Rc \geq -(n-1)\), and \(\Omega \subseteq M\) any bounded domain. If \(f \in C(\overline{\Omega})\) satisfies \(\Delta f \geq \delta > 0\) in the support sense, then \(f(x) \leq \max_{\partial \Omega}(f - \delta U(d(\cdot, x)))\).

Proof. \(\Delta(f - \delta U(d(\cdot, x))) \geq 0\), so the claim follows from the strong maximum principle. \(\square\)

Theorem 3. For any complete pointed Riemannian manifold \((M, g, p)\) with \(Rc \geq -(n-1)\), there exists \(\phi \in C^\infty(\overline{B_{R_2}(p)})\) such that \(\phi |_{B_{R_1}(p)} \equiv 1\) and \(|\nabla \phi| + |\Delta \phi| \leq c(n, R_1, R_2)\).

Proof. Let \(f \in C^\infty(A_{R_1, R_2}(p)) \cap C^0(\overline{A_{R_1, R_2}(p)})\) solve the boundary problem

\[
\begin{cases}
\Delta f = 1 & \text{in } A_{R_1, R_2}(p) \\
f = L_{R_2}(R_1) & \text{on } \partial B_{R_1}(p) \\
f = 0 & \text{on } \partial B_{R_2}(p)
\end{cases}
\]

Note that, since the boundary data is smooth, the existence of a solution will follow from the existence of a corresponding Dirichlet problem, which is well posed. Set 

\(b := L_{R_2}(R_1 + \eta_1)\) and \(a := L_{R_2}(R_1) - U(R_2 - R_1 - \eta_2) > L_{R_2}(R_1) - U(0) = 0\), and choose \(\psi \in C^\infty([0, 1])\) with \(\psi([-\infty, a]) \equiv 0\) and \(\psi(b, \infty) \equiv 1\). Here \(\eta_1, \eta_2\) are chosen sufficiently small so that \(a < b\). Since \(f \geq L_{R_2}(d(\cdot, p))\) on \(\partial A_{R_1, R_2}(p)\) and \(\Delta f = 1 \leq \Delta L_{R_2}(d(\cdot, p))\) in \(A_{R_1, R_2}(p)\), we have \(f(x) \geq L_{R_2}(d(x, p))\) for all \(x \in A_{R_1, R_2}(p)\). Since \(L_{R_2}\) is decreasing, this implies \(f \geq b\) on \(A_{R_1, R_1 + \eta_1}(p)\). In particular, \(\phi := \psi \circ f\) satisfies \(\phi = 1\) in a neighborhood of \(\partial B_{R_1}(p)\), so \(\phi\) can be extended smoothly to \(\overline{B_{R_2}(p)}\) by \(\phi |_{B_{R_2}(p)} \equiv 1\). The quantitative maximum principle gives

\[f(x) \leq \max_{\partial A_{R_1, R_2}(p)} (f - U(d(\cdot, x))),\]

but \(f - U \leq 0\) on \(\partial B_{R_2}(p)\), while \(f(x) \geq L_{R_2}(d(x, p)) \geq 0\), so the maximum occurs on \(\partial B_{R_1}(p)\). From this and the fact that \(U\) is increasing, we get

\[f(x) \leq L_{R_2}(R_1) - U(d(x, p) - R_1) < L_{R_2}(R_1) - U(R_2 - R_1 - \eta_2) = a\]

for all \(x \in A_{R_2 - \eta_2, R_2}(p)\), hence \(\phi |_{A_{R_2 - \eta_2, R_2}(p)} \equiv 0\). Thus

\[|\nabla \phi| + |\Delta \phi| = |(\psi' \circ f)\nabla f| + |(\psi'' \circ f)(\nabla f)^2| \leq C(R_1, R_2, \kappa)\]

by the Cheng-Yau gradient estimate. \(\square\)

Remark 4. If the distance function were smooth, and \(|\Delta r|\) bounded, we could use \(\eta(d(\cdot, p))\) as our cutoff for some \(\eta \in C^\infty([0, 1])\). Since this is not the case, we have to use a smooth approximation to a radial function on \(A_{R_1, R_2}(p)\), which is quantifiably large near \(\partial B_{R_1}(p)\) and quantifiably small near \(\partial B_{R_2}(p)\). It is tempting to use a harmonic approximation of some radial function, but this is made difficult by the geometry of annuli. In fact, there is a good lower comparison function \(L_{R_2}\) to wedge under a function, hence show it has some definite size near \(\partial B_{R_1}\), while there is no good upper comparison function to push the function down near \(\partial B_{R_2}(p)\). Thus the approximating function must be chosen so that it is uniformly subharmonic, so that it decreases by some fixed amount when it moves away from the inner boundary \(\partial B_{R_1}(p)\).
LINEAR FUNCTION IMPLIES METRIC PRODUCT

Suppose \( u \in C^\infty(M) \) satisfies \( \nabla^2 u = 0 \) and \( |\nabla u| = 1 \). Without loss of generality, \( N := u^{-1}(0) \neq \emptyset \), so we can define

\[
\psi : N \times \mathbb{R}, (x, t) \mapsto \varphi_t(x),
\]

where \( (\varphi_t) \) is the flow of \( \nabla u \), so that \( u(\varphi_t(x)) = t \). From \( L_{\nabla u} g = 0 \), we can compute directly that \( \psi^* g = dt^2 + g_N \). However, this construction is not stable with respect to “small” perturbations of \( u \), so we’d like to prove the fact that \( g \) is a metric product in a different way.

Choose \( x, z \in M \) with \( u(x) = u(z) \), and suppose \( w \in M \) satisfies \( d(w, z) = |u(w) - u(z)| \). Without loss of generality, assume \( u(z) = a > 0 = u(w) \), so that \( z = \varphi_a(w) \). Set \( \sigma(s) := \varphi_s(w) \), so that for almost every \( s \in [0, a] \), there exists a unique arclength minimizing geodesic \( \tau_s : [0, \ell(s)] \to M \) from \( x \) to \( \sigma(s) \), where \( \ell(s) := d(x, \sigma(s)) \). By the first variation formula, \( \ell'(s) = (\dot{\sigma}(s), \dot{\tau}_s(\ell(s))) = \langle \nabla u(\sigma(s)), \dot{\tau}_s(\ell(s)) \rangle \), so

\[
\frac{1}{2} \ell^2(u) - \frac{1}{2} \ell^2(0) = \int_0^a \langle \nabla u(\sigma(s)), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds
= \int_0^a \int_0^{\ell(s)} \langle \nabla u(\tau_s(\ell(s))), \dot{\tau}_s(\ell(s)) \rangle dt ds
= \int_0^a \int_0^{\ell(s)} \langle \nabla u(\tau_s(t)), \dot{\tau}_s(t) \rangle dt ds
= \int_0^a (u(\sigma(s)) - u(x)) ds
= \int_0^a (s - a) ds = -\frac{1}{2} a^2,
\]

where we used that \( t \mapsto \langle \nabla u(\tau_s(t)), \dot{\tau}_s(t) \rangle \) is constant since \( \nabla^2 u = 0 \). That is,

\[
d^2(x, w) = d^2(x, z) + d^2(w, z).
\]

Given any \( z \in u^{-1}(0) \), \( t \in \mathbb{R} \), we can find \( w \in M \) such that \( d(z, w) = |u(z) - u(w)| \) and \( u(z) = t \). In fact, just take \( w = \varphi_t(z) \). Define \( \psi(z, t) := w \), which coincides with the earlier definition of \( \psi \). We claim that \( \psi \) is a metric isometry. In fact, since \( t \mapsto \psi(y, t) \) is a minimal geodesic,

\[
d^2(\psi(x, t), \psi(y, s)) = d^2(\psi(y, t), \psi(y, s)) + d^2(\psi(y, t), \psi(x, t))
= |s - t|^2 + d^2(\psi(x, t), y) - d^2(\psi(y, t), y)
= |s - t|^2 + d^2(\psi(x, t), x) + d^2(y, x) - t^2
= |s - t|^2 + d^2(x, y).
\]

\( \delta \)-SPLITTING MAP IMPLIES ALMOST-METRIC PRODUCT

**Theorem.** For any \( \epsilon > 0 \), there exists \( \delta = \delta(n, \epsilon) > 0 \) such that, if \( \operatorname{Rc} \geq -(n-1)\delta \), \( r \in (0, 1] \), and if \( u : B_{10r}(p) \to \mathbb{R}^k \) is a \((k, \delta)\)-splitting map, then there exists a complete metric length space \((Z, d_Z, z^*)\) such that

\[
d_{G_H} \left( (B_r(p), d, p), (B_r(z^*, 0^k), d_Z \times \mathbb{R}^k, (z^*, 0^k)) \right) < \epsilon r.
\]

By rescaling, we assume \( r = 1 \) throughout.
Lemma 5. Suppose $x, z, w \in B_2(p)$ satisfy $|u(x) - u(z)| \leq \delta$ and $|u(z) - u(w)| - d(z, w) \leq \delta$. Then

$$|d^2(x, z) + d^2(z, w) - d^2(x, w)| \leq \Psi(\delta |n|).$$

Remark. Since $|u(z) - u(w)| \leq d(z, w) + \delta$, the the condition $|u(z) - u(w)| - d(z, w) \leq \delta$ is equivalent to

$$|u(z) - u(w)| \geq d(z, w) - \delta.$$

That is, the change in $u$ is almost maximal along any geodesic from $z$ to $w$, which is a way of measuring how close $z, w$ are to lying on the same line $\{z\} \times \mathbb{R}$.

Proof. First suppose $k = 1$. Without loss of generality, $d^2(x, w) \geq \Psi(\delta |n|)$. Set

$$h := |\nabla u|^2 + |\nabla u|^2 - 1,$$

so that the segment inequality and volume comparison give

$$\int_{B_2(p) \times B_2(p)} \mathcal{F}_h(x_1, x_2) d(\mu \otimes \mu)(x, y) \leq \int_{B_5(p)} h d\mu \leq C(n) \delta,$$

where $\mathcal{F}_h(x_1, x_2) = \int_0^1 \int_0^1 h(\gamma_{x_1, x_2}(t)) dt$, and $\gamma_{x_1, x_2}$ is the unique minimizing arclength geodesic from $x_1$ to $x_2$ for an open set of full measure in $B_2(p) \times B_2(p)$. Similarly,

$$\int_{B_2(p) \times B_2(p) \times B_2(p)} \mathcal{F}_{\mathcal{F}_n}(x_1, x_2, x_3) d(\mu \otimes \mu \otimes \mu)(x, y, z) \leq C(n) \int_{B_5(p) \times B_5(p)} \mathcal{F}_h(x_1, x_2) d(\mu \otimes \mu)(x_1, y) \leq C(n) \int_{B_5(p)} h d\mu \leq C(n) \delta.$$

Claim: There exist $\tilde{x}, \tilde{z}, \tilde{w} \in B_2(p)$ such that the following hold:

1. $d(\tilde{x}, x) + d(\tilde{z}, z) + d(\tilde{w}, w) + |\nabla u| |\tilde{u} - 1| \leq \Psi(\delta |n|)$,

2. $\sigma := \gamma_{\tilde{w}, \tilde{z}}$ is unique, and $\mathcal{F}_h(\tilde{w}, \tilde{z}) = \int_0^\delta h(\sigma(s)) ds \leq \Psi(\delta |n|)$, where $\tilde{d} := d(\tilde{w}, \tilde{z})$.

3. There exists an open subset $\tilde{U} \subseteq [0, \tilde{d}]$ of full measure such that, for all $s \in \tilde{U}$, the minimal geodesic $\tau_s : [0, \tilde{d}(s)] \to M$ from $\tilde{x}$ to $\sigma(s)$ is unique.

4. $\mathcal{F}_{\mathcal{F}_n}(\tilde{w}, \tilde{z}) = \int_0^\delta \int_0^\delta h(\tau_{\tilde{s}}(t)) dt ds \leq \Psi(\delta |n|)$.

In fact, by volume comparison and $\int_{B_5(p)} h d\mu \leq \Psi(\delta |n|)$, there is a subset of $\tilde{w} \in B_{\Psi(\delta |n|)}(w)$ of measure at least $c(n) |\Psi(\delta |n|) - 1|$. Thus, there is a subset of $\tilde{z} \in B_{\Psi(\delta |n|)}(z)$ of measure at least $c(n) |\Psi(\delta |n|) - 1|$. Then, there is a subset of $\tilde{x} \in B_{\Psi(\delta |n|)}(x)$ of measure at least $c(n) |\Psi(\delta |n|) - 1|$ such that (3), (4) hold. □

By composing with a translation, we can assume $u(z) > u(w) = 0$. Then, by (1) and $|\nabla u| \leq 1 + \delta$, we have $|u(\tilde{w})| \leq \Psi(\delta |n|)$ and $|u(\tilde{z}) - u(\tilde{w})| - \tilde{d} \leq \Psi(\delta |n|)$. In particular,

$$\int_0^\delta (|\nabla u(\sigma(s)), \sigma'(s)| - 1) ds \leq \Psi(\delta |n|),$$

but since $|\langle \nabla u, \sigma' \rangle| = 1 - \delta$, this implies

$$\int_0^\delta |\langle \nabla u(\sigma(s)), \sigma'(s) \rangle - 1| ds \leq \Psi(\delta |n|).$$
In particular,

$$|u(\sigma(t)) - u(\sigma(0)) - t| = \left| \int_0^t (\nabla u(\sigma(s)), \dot{\sigma}(s)) - 1 \right| ds \leq \Psi(\delta|n),$$

so $|u(\sigma(t)) - t| \leq \Psi(\delta|n)$. Also, since $\int_0^d \left| |\nabla u|^2 - 1 \right| (\sigma(s)) ds \leq \Psi(\delta|n)$,

$$\int_0^d |\nabla u(\sigma(s)) - \dot{\sigma}(s)| ds \leq 10 \left( \int_0^d |\nabla u(\sigma(s)) - \dot{\sigma}(s)|^2 ds \right)^{\frac{1}{2}} \leq \frac{10}{2} \left( \int_0^d (|\nabla u(\sigma(s))|^2 + 1 - 2\langle \nabla u(\sigma(s)), \dot{\sigma}(s) \rangle) ds \right)^{\frac{1}{2}} \leq \frac{10}{2} \left( \Psi(\delta|n) + 2 \int_0^d |\nabla u(\sigma(s)), \dot{\sigma}(s)| ds \right)^{\frac{1}{2}} \leq \Psi(\delta|n).$$

The first variation formula gives $\ell^p(s) = \langle \dot{\sigma}(s), \dot{\tau}_s(\ell(s)) \rangle$, so that

$$\frac{1}{2}(\ell^p(\tilde{d}) - \ell^p(0)) = \int_0^d \ell^p(s) \ell(s) ds = \int_0^d \langle \dot{\sigma}(s), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds = \int_0^d \langle \nabla u(\tau_s(\ell(s))), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds + \int_0^d \langle \dot{\sigma}(s) - \nabla u(\sigma(s)), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds = \int_0^d \int_0^{\ell(s)} \langle \nabla u(\tau_s(\ell(s))), \dot{\tau}_s(\ell(s)) \rangle dt ds + \int_0^d \langle \dot{\sigma}(s) - \nabla u(\sigma(s)), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds = \int_0^d \int_0^{\ell(s)} \langle \nabla u(\tau_s(t)), \dot{\tau}_s(t) \rangle dt ds + \int_0^d \langle \dot{\sigma}(s) - \nabla u(\sigma(s)), \dot{\tau}_s(\ell(s)) \rangle \ell(s) ds + \int_0^d \int_0^{\ell(s)} \nabla^2 u(\tau_s(r), \dot{\tau}_s(r)) dr dt ds.$$

However, we can estimate

$$\left| \int_0^d \int_0^{\ell(s)} \langle \nabla u(\tau_s(t)), \dot{\tau}_s(t) \rangle dt ds + \frac{1}{2} \tilde{d}^2 \right| = \left| \int_0^d \langle u(\sigma(s)) - u(\tilde{x}) \rangle ds + \frac{1}{2} \tilde{d}^2 \right| \leq \left| \int_0^d (s - \tilde{d}) ds + \frac{1}{2} \tilde{d}^2 \right| + \Psi(\delta|n) \leq \Psi(\delta|n),$$

so the claim follows by (1) – (4).
Now suppose $k \neq 1$. Suppose $x, z, w \in B_2(p)$ satisfy $|u(x) - u(z)| \leq \delta$ and $|u(z) - u(w)| - d(z, w) \leq \Psi(\delta|u)$. Then

$$u^* := \sum_{j=1}^{k} \frac{u_j(w) - u_j(z)}{|u(w) - u(z)|} u_j,$$

is a $C(n)\delta$-splitting function with $|u^*(w) - u^*(z)| = |u(w) - u(z)| \geq d(w, z) - \Psi(\delta|n)$ and $|u^*(x) - u^*(z)| \leq \delta$. Thus we can apply the previous lemma to get

$$|d^2(x, z) + d^2(z, w) - d^2(z, w)| \leq \Psi(\delta|n).$$

\qed