Integral inequalities

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This is a lecture note for the seminar on Cheeger-Colding theory on June 30. In this lecture, I’d like to talk about Cheeger-Colding’s segment inequality and its applications.

First we introduce Colding’s integral estimates. Let \( \mu \) be the Liouville measure on the spherical bundle \( \pi : SM = \{(p,v) : v \in T_p M, |v| = 1\} \rightarrow M \) defined as the product of Riemannian volume measure and the normalized spherical measure (of total mass 1).

Let \( GF_t \) be the geodesic flow on \( SM \), i.e.,

\[
GF_t(v) = \exp_{\pi(v)}(tv), \quad v \in SM, t \in \mathbb{R}.
\]

It is well-known that \( GF_t \) preserves the Liouville measure:

\[
GF_t \mu = \mu, \quad (GF_t \mu)(A) = \mu(GF_t^{-1}(A)) = \mu(GF_t^{-1}(A)).
\]

Or for any \( A \subseteq SM \) and integrable \( f \),

\[
\int_A f(v) d\mu(v) = \int_A f(v) d(GF_t \mu)(v) = \int_{GF_t^{-1}(A)} f(GF_t(v)) d\mu(v).
\]

**Theorem 1.** For any \( r, l > 0, \lambda \in \mathbb{R} \), and any \( f \in C^\infty(M) \),

\[
\int_{SB_r(p)} \left| \langle \nabla f, v \rangle - \frac{f(\gamma_v(l)) - f(\gamma_v(0))}{l} - \lambda \right| d\nu \leq l \int_{B_{r+l}(p)} |\nabla^2 f + 2\lambda g|.
\]

**Proof.** Write \( h(t) = f(\gamma_v(t)) \).

\[
\langle \nabla f, v \rangle - \frac{f(\gamma_v(l)) - f(\gamma_v(0))}{l} - \lambda l = -\frac{1}{l} \int_0^l h'(t) - h'(0) + 2\lambda t dt = -\frac{1}{l} \int_0^l dt \int_0^t h''(s) + 2\lambda ds
\]

\[
= -\frac{1}{l} \int_0^l ds \int_0^t h''(s) + 2\lambda dt.
\]

Then

\[
\left| \langle \nabla f, v \rangle - \frac{f(\gamma_v(l)) - f(\gamma_v(0))}{l} - \lambda \right| \leq \int_0^l |\nabla^2 f + 2\lambda g(\gamma_v(t))| dt.
\]
Integrating on $SB_r(p)$,
\[
\int_{SB_r(p)} \left| \langle \nabla f, v \rangle - \frac{f(\gamma_+(l)) - f(\gamma_+(0))}{l} \right| dv 
\leq \int_{SB_r(p)} \int_0^l |\nabla^2 f + 2\lambda g| dv \, dt = \int_0^l dt \int_{SB_r(p)} |\nabla^2 f + 2\lambda g| dv 
\leq l \int_{B_{r+1}(p)} |\nabla^2 f + 2\lambda g| dx.
\]

The following integral estimate plays an important role in Cheeger-Colding theory. Write
\[
F_f(x_1, x_2) = \inf_{\gamma} \int_0^{[x_1, x_2]} f(\gamma(s)) \, ds,
\]
where the infimum is taken over all minimizing normal geodesic $\gamma$ joining $x_1$ and $x_2$.

**Theorem 2.** Suppose $Rc \geq -(n-1)\lambda$. For any $r \leq R$, $A_1, A_2 \subseteq B_r(p)$ and nonnegative $f \in L^1(B_{2R}(p))$,
\[
\int_{A_1} dx_1 \int_{A_2} F_f(x_1, x_2) \, dx_2 \leq C(n, \lambda, R) r \cdot (|A_1| + |A_2|) \int_{B_{2R}(p)} f.
\]

**Proof.** Consider the set $W \subseteq A_1 \times A_2$ consisting of $(x_1, x_2)$ so that there is a unique minimizing geodesic on $M$ connecting them. It is standard that $W$ has full measure in $A_1 \times A_2$. So it suffices to consider the integral on $W$. By a little abuse of notations, we still write $A_1, A_2$ omitting the points outside $W$.

Choose $x_1 \in A_1$. Let $J(t, \theta) dt \wedge d\theta$ be the volume form of the spherical geodesic coordinate centered at $x_1$ so that $J(t, \theta) d\theta$ is the volume (area) form of the geodesic sphere $\partial B_t(x_1)$. By the Ricci lower bound, we know
\[
\frac{J(t, \theta)}{sn^{n-1}_\lambda(t)}.
\]

So it suffices to consider one term above by symmetry.

Fix $x_1 \in A_1$. Let $J(t, \theta) dt \wedge d\theta$ be the volume form of the spherical geodesic coordinate centered at $x_1$ so that $J(t, \theta) d\theta$ is the volume (area) form of the geodesic sphere $\partial B_t(x_1)$. By the Ricci lower bound, we know
\[
\frac{J(t, \theta)}{sn^{n-1}_\lambda(t)}.
\]

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Let \( J(t, \theta) \leq \frac{\text{sn}^{-1}(t)}{\text{sn}^{-1}(s)} \leq \frac{\text{sn}^{-1}(t)}{\text{sn}^{-1}(t/2)} \leq C(n, \lambda, R) \).

Let \( V_2 \subseteq T_{x_1}M \) such that \( \exp_{x_1}(U_2) = A_2 \). For \( t \in [0, 2r] \), set \( S(r) = \{ \theta \in S^{n-1} : r\theta \in U_2 \} \). Then

\[
\int_{A_2} \int_0^{\vert x_1 x_2 \vert} f(\gamma_{x_1, x_2}(s)) \, ds \, dx_2
\]

\[
= \int_{U_2} J(t, \theta) \, dt \, d\theta \int_{t/2}^t f(\exp_{x_1}(s\theta)) \, ds
\]

\[
\leq C(n, \lambda, R) \int_0^{2r} dt \int_{S^{n-1}} d\theta \int_{t/2}^t f(\exp_{x_1}(s\theta)) J(s, \theta) \, ds
\]

\[
\leq C r \int_{B_{2r}(0)} f(\exp_{x_1}(s\theta)) J(s, \theta) \, ds \, d\theta
\]

As an application, we prove the following Poincare inequality of Dirichlet type: on a manifold \((M^n, g)\) with \(\text{Rc} \geq -(n-1)\lambda, \partial B_{3r}(x)\) nonempty,

\[
\int_{B_r(x)} \vert f \vert^p \leq C(n, p, \lambda) \cdot r^p \int_{B_r(x)} \vert \nabla f \vert^p,
\]

for any \( f \in C_c^\infty(B_r(x)) \), \( r \in (0, 1] \), \( p \in [1, n] \).

Proof. Let \( A_1 = B_r(x) \). Pick \( y \in \partial B_{3r}(x) \). Set \( A_2 = B_r(y) \). Let \( x_j \in A_j \). Note that for any minimizing geodesic \( \gamma : [0, l] \rightarrow M \) connecting \( x_1, x_2 \),

\[
\int_{B_r(y)} \int_{B_r(x)} \vert f \vert^p(x_1) \, dx_1 \leq \int_{B_r(x) \times B_r(y)} \mathcal{F}_{\vert \nabla f \vert} \leq C r \vert B_r(x) \vert \int_{B_r(x)} \vert \nabla f \vert^p.
\]

Then

\[
\int_{B_r(x)} \vert f \vert^p \leq C p r \int_{B_r(x)} \vert f \vert^{p-1} \vert \nabla f \vert \leq C r \left( \int_{B_r(x)} \vert f \vert^p \right)^{1-1/p} \left( \int_{B_r(x)} \vert \nabla f \vert^p \right)^{1/p}.
\]

\[\square\]
Poincare inequality of Neumann type: for $r \leq 1$ and smooth $f$ on $M$,

$$\int_{B_r(x)} |f - f_{B_r}|^p \leq C(n, p, \lambda) r^p \int_{B_{2r}(x)} |\nabla f|^p.$$

**Proof.** Note that

$$|f(y) - f(z)|^p \leq \left( \int_0^1 |\nabla f(\gamma(t))| \right)^p \leq l^{1-1/p} \int_0^l |\nabla f|^p = |yz|^{1-1/p} \mathcal{F}_{|\nabla f|^p}(y, z).$$

$$\int_{B_r(x)} |f - f_{B_r}|^p = \int_{B_r(x)} \left( \int_{B_r(x)} f(y) - f(z) \, dz \right)^p \, dy$$

$$\leq \int_{B_r(x)} \int_{B_r(x)} |f(y) - f(z)|^p \, dz \, dy$$

$$\leq (2r)^{p-1} \int_{B_r(x)} \mathcal{F}_{|\nabla f|^p} \, dz \, dy$$

$$\leq C \cdot r^p \int_{B_{2r}(x)} |\nabla f|^p.$$

\[\square\]

**Theorem 3 (Sobolev).** Suppose $\text{Re} \geq -(n-1)$. Then for any $\epsilon \in (0, 1), p \geq 1$,

$$\left\{ \frac{1}{q} \left( \frac{1}{q} \right)^{-1/q} \right\}^{1/p} \leq C(n, p, \epsilon) \cdot r \left\{ \frac{1}{p} \left( \frac{1}{p} \right)^{-1/p} \right\}^{1/p},$$

for any $r \in (0, 1], f \in W^{1,p}(B_r(x))$, where

$$f_B = \int_B f, \quad q = \frac{np}{n - (1-\epsilon)p}.$$

**Proof.** WLOG assume $f_{B_r(x)} = 0$. Consider

$$A_t \equiv \{|f| > t\} \cap B_r(x).$$

For any Lebesgue point $y \in A_t$ of $f$, consider

$$B_0 = B_r(x), \quad B_j = B_{2^{-j}r}(y).$$

Then

$$t < |f(y) - f_{B_0}| \leq \sum_{j \geq 1} |f_{B_j} - f_{B_{j-1}}|$$

$$\leq \sum_{j \geq 1} \int_{B_{j-1}} |f - f_{B_j}| \leq C(n) \sum_{j \geq 1} \int_{2B_j} |f - f_{B_j}|$$

$$\leq C(n) \cdot r \int_{4B_j} |\nabla f|,$$
where \( r_j = 2^{-j}r \). Write
\[
t = (1 - 2^{-e})t \sum_{j \geq 0} 2^{-j} = c(n, \epsilon)tr^{-\epsilon} \sum_{j \geq 0} r_j^\epsilon.
\]
Then
\[
\sum_{j \geq 1} r_j \int_{4B_j} |\nabla f| \geq c(n, \epsilon)tr^{-\epsilon} \sum_{j \geq 1} r_j^\epsilon.
\]
It follows that there exists some \( j \geq 1 \), such that
\[
r_j \int_{4B_j} |\nabla f| \geq c(n, \epsilon)tr^{-\epsilon}r_j^\epsilon.
\]
Let \( B_y = 4B_j \) be the ball.
\[
\int_{B_y} |\nabla f| \geq c(n, \epsilon)(r/r_j)^{1-\epsilon}t/r |B_y| \geq c(n, \epsilon) \left( \frac{|B_r(x)|}{|B_y|} \right)^{\frac{1-\epsilon}{\epsilon}} |B_y|t/r.
\]
By Vitali covering, \( A_\epsilon \) can be covered by \( \{5B_y_j\} \) so that \( \{B_y_j\} \) are pairwise disjoint. Write
\[
\mu = 1 - \frac{1 - \epsilon}{n} \in (0, 1).
\]
Then
\[
|B_y|^\mu \leq C(n, \epsilon)rt^{-1} |B_r(x)|^{-\frac{1-\epsilon}{\epsilon}} \int_{B_y} |\nabla f|.
\]
It follows that
\[
|A_i|^\mu \leq \left\{ \sum_j |5B_y_j| \right\}^\mu \leq C(n, \epsilon) \sum_j |B_y|^\mu
\]
\[
\leq C(n, \epsilon)rt^{-1} |B_r(x)|^{-\frac{1-\epsilon}{\epsilon}} \sum_j \int_{B_y_j} |\nabla f|
\]
\[
\leq C(n, \epsilon)rt^{-1} |B_r(x)|^\mu \int_{B_r(x)} |\nabla f|.
\]
Let \( q = \frac{1}{1 - \frac{1-\epsilon}{n}} < 1/\mu \). Then for \( s > 0 \) TBD,
\[
\int_{B_r(x)} |f|^q = \frac{1}{q|B_r(x)|} \int_0^\infty t^{q-1} |A_t|dt
\]
\[
\leq C(n, \epsilon)s^q + C(n, \epsilon) \left( r \cdot \int_{B_r(x)} |\nabla f| \right)^{1/\mu} \int_s^\infty t^{q-1-1/\mu} dt
\]
\[
\leq C(n, \epsilon) \left\{ s^q + s^{q-1/\mu} \left( r \cdot \int_{B_r(x)} |\nabla f| \right)^{1/\mu} \right\} =: C(s^q + As^{q-1/\mu}).
\]
The optimal $s$ on the RHS is

$$s = \left\{ \frac{(1 - q + 1/\mu)A}{q} \right\}^\mu$$

and thus

$$\left\{ \int_{B_r(x)} |f|^q \right\}^{1/q} \leq C(n, \epsilon)A^\mu = C(n, \epsilon) \cdot r \int_{B_r(x)} |\nabla f|.$$