Exercise 2.1.2. Verify that \((X, d_L)\) is a metric space.

Solution. Let \(X\) be a Hausdorff topological space and

\[
(X, A, L : A \to \mathbb{R}_+ \cup \{\infty\})
\]

be a length structure on \(X\). This means that conditions (1)-(3) on p. 26 and (1)-(4) on p. 27 are satisfied. By definition, if \(x, y \in X\)

\[
d_L (x, y) = \inf \{ L(\gamma) ; \gamma : [a, b] \to X, \gamma (a) = x, \gamma (b) = y \}.
\]

For every \(\gamma \in A, L(\gamma) \in \mathbb{R}_+ \cup \{\infty\}\) so \(\forall x, y \in X, d_L (x, y) \in \mathbb{R}_+ \cup \{\infty\}\). By (1) on p. 26 "closed under restrictions", if \(x \in X\), then the point path \(\gamma : [a, a] \to X\) defined by \(\gamma (a) = x\) is contained in \(A\). Then (see p. 27) \(L (\gamma, a, a) = 0\). Hence \(d_L (x, x) = 0\). Now suppose \(x, y \in X, x \neq y\). Since \(X\) is Hausdorff, there exists a neighborhood \(U_x\) of \(x\) such that \(y \in X \setminus U_x\). By property (4) "agree with the topology",

\[
d_L (y, x) = d_L (x, y) \text{ follows from property (3) "closed under linear reparametrizations".}
\]

The triangle inequality follows from (2) "closed under concatenations".