Problem 1 (2.1.9). Prove that a length space is locally path-connected: every neighborhood of any point contains a smaller neighborhood which is path-connected.

Proof. Let $X$ be a length space with length function $L$. The choice of path-connected neighborhood here is really easy: for some open neighborhood about a point $x \in X$, there is some $\eta$–ball $B(x, \eta)$ (in the topology of the length space) located inside the neighborhood. We claim this ball is path-connected.

Let $y \in B(x, \eta)$ and say $d_L(x, y) = \delta$ (recalling that $d_L$ is the metric induced by $L$). Then let $0 < \epsilon < 2(\eta - \delta)$, and choose a path $\gamma_\epsilon$ from $x$ to $y$ such that $L(\gamma_\epsilon) < \delta + \epsilon$. We show that $\gamma_\epsilon$ lies entirely in $B(x, \eta)$ (hence showing $B(x, \eta)$ is path-connected). Assume for contradiction that $\gamma_\epsilon$ lies outside $B(x, \eta)$, that is, if $\gamma_\epsilon$ is defined on $[a, b]$ then there exists some $c \in (a, b)$ such that $\gamma_\epsilon(c) \notin B(x, \eta)$. Note that

$$L(\gamma_\epsilon) = L(\gamma_\epsilon |_ {[a, c]}) + L(\gamma_\epsilon |_ {[c, b]}),$$

and

$$L(\gamma_\epsilon |_ {[a, c]}) \geq d_L(x, \gamma_\epsilon(c)) \geq \eta \quad L(\gamma_\epsilon |_ {[c, b]}) \geq d_L(y, \gamma_\epsilon(c)) \geq \eta - \delta$$

by the metric structure of $L$.

Thus we conclude that $L(\gamma_\epsilon) \geq 2\eta - \delta$, but by assumption $L(\gamma_\epsilon) < 2\eta - \delta$, a contradiction. Thus we see that $\gamma_\epsilon$ must lie entirely inside $B(x, \eta)$, and as our choice of $y$ was arbitrary this shows that $B(x, \eta)$ is path-connected. Hence $X$ is locally path-connected. \hfill \Box