Corollary 3.1.2. Consider two intrinsic metrics $d_1$ and $d_2$ defined on the same set $X$ and inducing the same topology. Assume that every point $x \in X$ has a neighborhood $U_x$ such that the restrictions of the metrics to this neighborhood coincide: for every $p, q \in U_x$, $d_1(p, q) = d_2(p, q)$. Then $d_1 = d_2$.

Exercise 3.1.3. Prove the corollary.

Solution. p. 28, Definition 2.1.6: A metric that can be obtained as the distance function associated to a length structure is called an intrinsic metric. Let $(X, A_1, L_1 : A_1 \to \mathbb{R}_+ \cup \{\infty\})$ and $(X, A_2, L_2 : A_2 \to \mathbb{R}_+ \cup \{\infty\})$ be length structures on $X$ with associated distance functions $d_{L_1} = d_1$ and $d_{L_2} = d_2$ as in the assumption of the corollary. We shall show that for every $p, q \in X$, $d_1(p, q) \geq d_2(p, q)$ (by symmetry $d_2(p, q) \geq d_1(p, q)$ so we get equality). It suffices to show for every $p, q \in X$ where $U_x$ are the sets in the hypothesis of the corollary.

Claim. There exist $a = c_0 < c_1 < c_2 < \cdots < c_k = b$ such that for every $i = 1, \ldots, k$,
$$
\gamma([c_{i-1}, c_i]) \subset U_{x_i}
$$
for some $x_i \in X$ and where $U_{x_i}$ are the sets in the hypothesis of the corollary. In particular, $\gamma(c_{i-1}), \gamma(c_i) \in U_{x_i}$.

Assuming the claim, we have

$$
L_1(\gamma) = \sum_{i=1}^k L(\gamma|_{[c_{i-1}, c_i]}) \\
\geq \sum_{i=1}^k d_1(\gamma(c_{i-1}), \gamma(c_i)) \\
= \sum_{i=1}^k d_2(\gamma(c_{i-1}), \gamma(c_i)) \\
\geq d_2(p, q).
$$

To prove the claim we note that since $\gamma$ is continuous, for every $c \in [a, b]$ there exists $\varepsilon(c) > 0$ such that $\gamma((c - \varepsilon(c), c + \varepsilon(c)) \cap [a, b]) \subset U_{\gamma(c)}$. Since $[a, b]$ is compact, there exist $a = d_0 < d_1 < d_2 < \cdots < d_k = b$ such that $\gamma((d_i - \varepsilon(d_i), d_i + \varepsilon(d_i)) \cap [a, b]) \subset U_{\gamma(d_i)}$. Choose $c_0 = a$, $c_k = b$ and $c_i \in (d_i - \varepsilon(d_i), d_i + \varepsilon(d_i))$ increasing for $i = 1, \ldots, k - 1$.

(Check to make sure that this completes the solution of the exercise. I haven’t carefully checked that this indeed finishes the argument.)