Problem. 3.4.5 Verify that \((C(X), d_\infty)\) is a metric space and that \(d_\infty(d(x, -), d(y, -)) = d(x, y)\)

Answer. I claim, \(C(X)\) is a normed vector space with \(d_\infty\) defined by the norm. Since multiplication and addition are continuous for functions \(\mathbb{R} \to \mathbb{R}\), it follows immediately that \(C(X)\) is an \(\mathbb{R}\)-vector space. Define \(|f|_\infty = \sup_{x \in X} |f(x)|\). This satisfies all of the properties of a norm,

If \(f \neq 0\) then \(\exists x \in X\) such that \(|f(x)| > 0 \Rightarrow |f| > 0\). If \(f\) is equivalent to 0 then obviously \(|f| = 0\)

\[|tf|_\infty = \sup |tf(x)| = \sup |t||f(x)| = |t| \sup |f(x)| = |t||f|_\infty\]

\[|f + g| = \sup |f(x) + g(x)| \leq \sup (|f(x)| + |g(x)|) \leq |f| + |g|

So we have a norm, thus a metric.

Claim \(d_\infty(d(x, -), d(y, -)) = d(x, y)\). This is the same as showing, \(\sup_{z \in X} |d(x, z) - d(z, y)| = d(x, y)\).

The triangle inequality implies \(d(x, z) \leq d(x, y) + d(y, z)\) which implies \(|d(x, z) - d(y, z)| \leq d(x, y)\) which implies \(\sup_{z \in X} |d(x, z) - d(y, z)| \leq d(x, y)\).

Now take \(z = x\), then \(|d(x, y)| = |d(x, x) - d(x, y)| \leq \sup_{z} |d(x, z) - d(z, y)|.\)