Let \( X \) be a complete metric space and \( Y \) be its closed subset. Assume that for any \( \epsilon \), \( Y \) has a finite \( \epsilon \)-net. Then \( Y \) is compact.

Proof: Argument by contradiction. Let \( U \) be an open covering and \( \{y_1^n, y_2^n, \ldots, y_k^n\} \) be a finite \( 2^{-n} \)-net of \( Y \).

Assume \( Y \) cannot be covered by any finite open sets of \( U \). Because \( Y \subset \bigcup B_{1/2}(y_i^1) \), which is finite, without loss of generality, we may assume the first one \( B_{1/2}(y_1^1) \) cannot be covered by any finite open sets. By the assumption, there are finite many balls so that \( B_{1/2}(y_1^1) \subset \bigcup B_{1/4}(y_i^2) \). Among the balls satisfying \( B_{1/4}(y_2^2) \cap B_{1/2}(y_1^1) \neq \phi \), there must be one, we may assume \( B_{1/4}(y_2^2) \) cannot be covered by any finite open sets.

Repeating this process, we get a sequence of balls \( B_{2^{-n}}(y_1^n) \), each of which cannot be covered by any finite open sets of \( U \). By the triangle inequalities we know that \( d(y_1^n, y_m^n) < 2/2^n \) for any \( n < m \), so \( \{y_i^n\} \) is a Cauchy sequence and it has a limit \( y \in Y \).

\( \exists U \in U \) such that \( y \in U \). There is an \( r \) small enough so that \( B_r(y) \subset U \). On the other hand, since \( d(y, y_1^n) \leq 2/2^n \), we know that \( B_{2^{-n}}(y_1^n) \subset B_r(y) \) when \( 2^{-n} < \frac{r}{4} \), . Therefore \( B_{2^{-n}}(y_1^n) \subset U \). But this is impossible by its construction.