Coherification of Equivalence in Gray-Categories (Draft)

Ben Hummon

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1 Introduction

In this paper, I discuss coherence relations for an equivalence of objects in a Gray-category by using diagrammatic arguments. We begin with a warm-up by working the analogous result for equivalence in a 2-category. Suppose we are given data observing an equivalence $f\colon X \simeq Y$ of objects in a 2-category. This data may not satisfy all the relations that we might hope for. Indeed, we might wish $1 \to f^{-1} \circ f$ and $f \circ f^{-1} \to 1$ be the unit and counit of an adjunction. Using string diagrams, such a rule corresponds to a "topologically reasonable" relation of pulling a string tight. By rechoosing some of the data for an equivalence, we can canonically improve, or *coherify*, the equivalence so that its data also satisfies these topological relations. In fact, the calculus of string diagrams makes it straightforward to solve for the fully coherent data in terms of the original data.

We'll be using 2-categories and Gray-categories instead of their fully weak counterparts. Any (weak) bicategory is (bi-)equivalent to a strict 2-category. We'll therefore be content to use string diagrams for 2-categories. Such string diagrams are made of strings embedded in a 2-dimensional square. By adding decorations, one could have diagrams that instead evaluate in a bicategory. Things get more complicated for tricategories. While it is not the case that any tricategory is (tri-)equivalent to a strict 3-category, Gordon, Power, and Street showed [6] that any tricategory is (tri-)equivalent to a (semi-strict) Gray-category. Thus, we'll use diagrams to encode composition in a Gray-category rather than in a tricategory. In principle, we could decorate these diagrams to work for tricategories, but this would do more to obscure than illuminate.

This paper is part of ongoing work on surface diagrams. In my as of yet unreleased thesis I provide a formalism for using surface diagrams to represent compositions in a Gray-category. A surface diagram consists of a stratified surface with 2D foams, 1D seams, and 0D nodes, embedded in a cube. As the surface may be separating in 3-space, we'll also think about 3D regions between surface. We label strata according to codimension with morphisms from a fixed Gray-category. So objects label regions, 1-morphisms label foams, and so on. At all but finitely many heights, we may horizontally slice a surface diagram to get an (ordered) string diagram. Each exceptional height corresponds to either a node (encoding a 3-morphism) or to seams swapping order (encoding an interchange).

We insist that our surface diagrams satisfy progressivity requirements: seams may never travel horizontally and foams must never be tangent to the back-front direction. At first glance, these rules might seem quite restrictive. Indeed, the only type of non-stratified surface allowed is one parallel to the back face of the cube. Let's consider the string diagram case for comparison. A progressive string diagram is one in which the string never travels horizontally. As long as a non-progressive string diagram only has isolated critical points, we can regard it as a progressive stratified diagram by taking the critical points to be in the 0-stratum. Similarly, most non-progressive non-stratified surfaces may be reinterpreted as progressive stratified diagrams. I've chosen directional conventions for surface diagrams so that a surface's fold lines correspond to 1-strata and the critical points in the fold lines correspond to 0-strata.

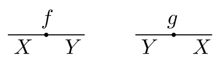
The coherification result for 2-categories is well known among experts. I begin with it to review definitions and because it figures into the result for Gray-categories in multiple ways. Nick Gurski [7] and Steve Lack [13] have each worked on similar coherence results for tricategories. This work was done independently and was used to sharpen the author's intuition for surface diagrams. It may be regarded as an advertisement for the value of visual intuition when working in higher categories.

I'd like to thank Michael Shulman and Justin Roberts for their many illuminating discussions. Also, many thanks to John Foley and Noel Dwyer for all their talking math and support.

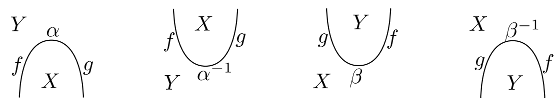
2 Equivalence in 2-Categories

We will use a fairly standard notation borrowed from tensor categories: our horizontal composition (⊗) goes from left to right and vertical composition goes up.

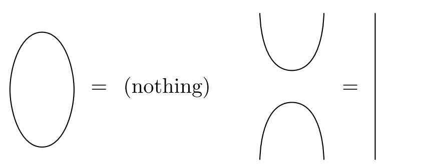
Definition. In a 2-category, an (*incoherent*) *equivalence* of objects $\overset{X}{\cdot}$ and $\overset{Y}{\cdot}$ consists of 1-morphisms



as well as 2-morphisms

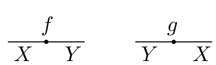


which satisfy inverse relations:

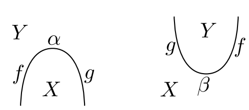


We think of f and g as allowing us to compare X and Y. We want f and g to essentially be inverse. That is, we want to be able to compare $f \circ g$ with 1_X and $g \circ f$ with 1_Y. This is where we get our 2-morphisms $\alpha, \alpha^{-1}, \beta$, and β^{-1}. We would like to be able to work with f and g as if they were actually inverse. We might worry that the two ways of making a cancellation $fgf \cong f$ could be composed to get a non-trivial automorphism of f. Indeed, we include the modifier "incoherent" in our definition because we do not exclude this possibility.

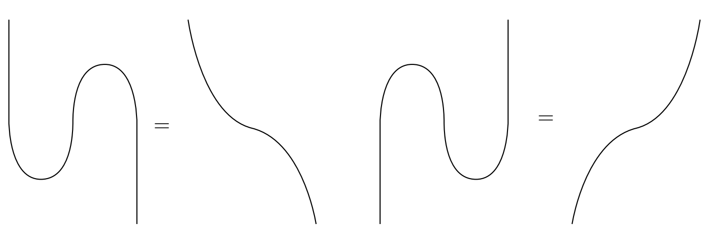
Definition. In a 2-category, a pair of 1-morphisms



are *adjoint* if there are 2-morphisms



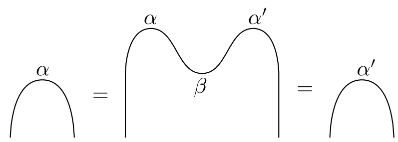
satisfying *zig-zag* relations



We may say that (g, f) are an adjoint pair with *unit* β and *counit* α. Note that this definition is not symmetric in f and g. One should look to the unit for the left adjoint to be on the left and the right adjoint to be on the right. This definition is meant to generalize the definition of adjoint functors in the 2-category of categories, functors, and natural transformations.

Lemma. In an adjunction, the unit and counit determine each other.

Proof. Suppose both α and α' satisfy the zig-zag relations with β. Then



∎

Definition. The previous lemma allows us to define an exponent $\sim$ which swaps between corresponding units and counits.

$$\alpha^{\sim} = \beta \quad \text{and} \quad \beta^{\sim} = \alpha.$$

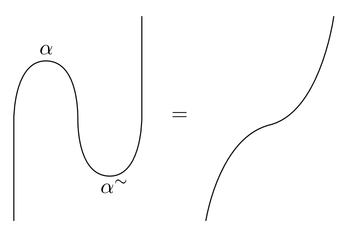
This should be thought of as playing a similar a role to the exponent -1 for inversion.

Definition. A *coherent equivalence* is an equivalence in which the 2-morphisms satisfy all four possible zig-zag relations:

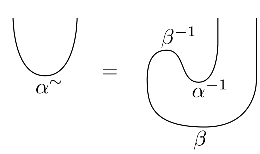
- β and α demonstrate (g, f) as an adjoint pair;
- α^{-1} and β^{-1} demonstrate (f, g) as an adjoint pair.

Theorem (equivalence coherification)**.** Given an incoherent equivalence as defined above, we may rechoose β and β^{-1} so as to get a coherent equivalence. Moreover, there is a unique such choice.

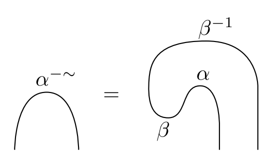
Proof. The reader is encouraged at this point to ignore the following and discover a proof for herself. To start, one may solve



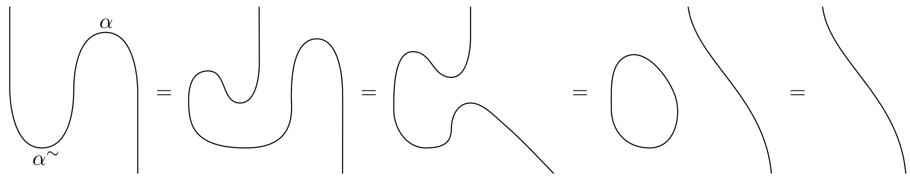
for $\alpha^{\sim}$ using only the inverse relations. We choose



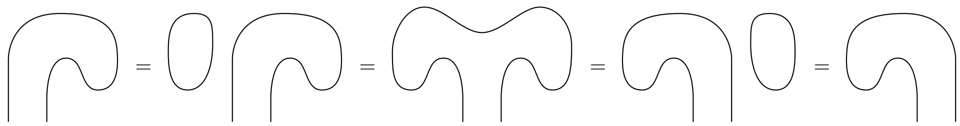
and



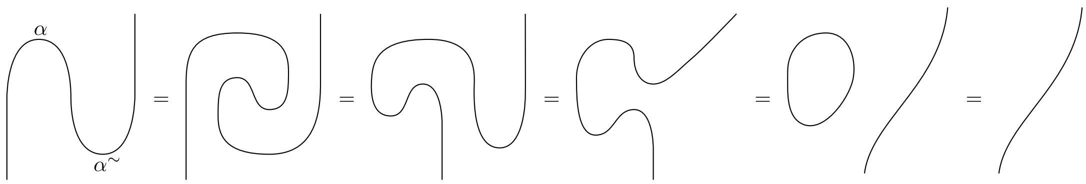
We will demonstrate $(\alpha, \alpha^{-1}, \alpha^{\sim}, \alpha^{-\sim})$ give a coherent equivalence. It is straightforward to see that $\alpha^{\sim}$ and $\alpha^{-\sim}$ are inverse. We must show that our data satisfy the four zig-zag relations. One of the four may be seen by the following:



Next, notice

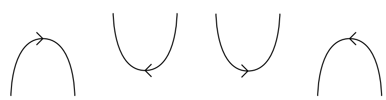


which allows us to show a second zig-zag relation:



The remaining two zig-zag relations are can be proved by using arguments whose diagrams are top-bottom reflections of the given ones. Uniqueness follows from uniqueness of inverses and the previous unit/counit uniqueness lemma. □

To make labels unnecessary, we can put orientations on the data of a coherent equivalence:



Then we just make a convention, say, X is to the right and Y is to the left as we travel along the string.

Once we have a coherent equivalence, it is easy to work with the 2-morphisms making defining such an equivalence. One can prove, in a certain Morse-theoretic sense, that the zig-zag relations generate all topological relations of strings in the plane. So one is free to manipulate any composition of coherent equivalence data using topological relations without changing the 2-morphism being described.

Application: Morita Equivalence of Rings

Let's take a quick digression to discuss the definitions and results of the previous section for the bicategory, Rng, whose 0-morphisms are rings, 1-morphisms are bimodules, and 2-morphisms are linear maps. Horizontal composition of bimodules is given by taking tensor product. In this category, incoherent equivalence is given the name *Morita equivalence*. The definition of adjoint pair gives the usual (sided) definition of dual bimodules.

Example. We may think of column vectors $\mathrm{Mat}_{n,1}(R)$ as forming an (R,R)-bimodule by using left and right scalar multiplication. We can similarly get a bimodule of row vectors, $\mathrm{Mat}_{1,n}(R)$. We can recognize $(\mathrm{Mat}_{1,n}(R), \mathrm{Mat}_{n,1}(R))$ as a dual pair with the maps

$$R \longrightarrow \mathrm{Mat}_{1,n} \otimes_R \mathrm{Mat}_{n,1}$$
$$r \mapsto \sum_i re^i \otimes e_i$$
$$\mathrm{Mat}_{n,1} \otimes_R \mathrm{Mat}_{1,n} \longrightarrow R$$

multiply then take trace.

We can also recognize $(\mathrm{Mat}_{n,1}(R), \mathrm{Mat}_{1,n}(R))$ as a dual pair with the maps

$$\begin{aligned} &R \longrightarrow \mathrm{Mat}_{n,1} \otimes_R \mathrm{Mat}_{1,n} \\ &r \mapsto \sum_i r e_i \otimes e^i \\ &\mathrm{Mat}_{1,n} \otimes_R \mathrm{Mat}_{n,1} \longrightarrow R \\ &\text{dot product.} \end{aligned}$$

It's worth noting, though, that these data don't satisfy the inverse relations of a coherent equivalence: they're off by a factor of n. So in $\mathbb{Z}$, or even worse, $\mathbb{Z}/n\mathbb{Z}$, this ambidextrous adjunction is quite far from a coherent equivalence.

Example. We may also think of column vectors $\mathrm{Mat}_{n,1}(R)$ as forming a $(\mathrm{Mat}_{n,n}(R), R)$-bimodule by using left and right matrix multiplication. Similarly, row vectors $\mathrm{Mat}_{1,n}(R)$ form an $(R, \mathrm{Mat}_{n,n}(R))$-bimodule. The matrix multiplication maps,

$$\begin{aligned} \mathrm{Mat}_{1,n} \otimes_{\mathrm{Mat}_{n,n}} \mathrm{Mat}_{n,1} &\longrightarrow R \\ \mathrm{Mat}_{n,1} \otimes_R \mathrm{Mat}_{1,n} &\longrightarrow \mathrm{Mat}_{n,n} \end{aligned}$$

are both invertible. To invert the first, we rely heavily on the $\mathrm{Mat}_{n,n}$-middle linearity. The inverse of the second can be thought of as, "regard a matrix as a sum of rank 1 matrices."

These maps give us the data for a coherent equivalence which demonstrates that any ring R is Morita equivalent to its ring of square matrices, $\mathrm{Mat}_{n,n}(R)$. It follows that the rings of $m \times m$ and $n \times n$ matrices are Morita equivalent. In fact, one can directly demonstrate this equivalence by considering the bimodule $\mathrm{Mat}_{m,n}(R)$ and its dual, $\mathrm{Mat}_{n,m}(R)$.

In the previous example, our choice of maps

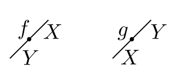
$$\begin{aligned} \alpha\colon \mathrm{Mat}_{1,n} \otimes_{\mathrm{Mat}_{n,n}} \mathrm{Mat}_{n,1} &\longrightarrow R \\ \beta^{-1}\colon \mathrm{Mat}_{n,1} \otimes_R \mathrm{Mat}_{1,n} &\longrightarrow \mathrm{Mat}_{n,n} \end{aligned}$$

as matrix multiplication gave a coherent equivalence of rings. We can fiddle with α as follows: given $r, s \in R$ invertible, we define $\alpha'(x) = r(\alpha(x))s$. This α' is invertible. It will, however, usually only demonstrate an incoherent equivalence with β. We can similarly fiddle with β by left and right multiplying by invertible matrices. Coherification improves these less-than-wonderful choices of α' and β' by normalizing one using the other.

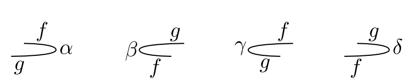
3 Equivalence in Gray-Categories

Semi-Coherification

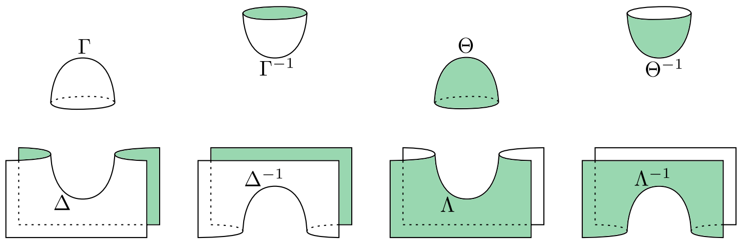
Definition. An *incoherent equivalence* in a Gray-category between 0-morphisms X and Y consists of 1-morphisms



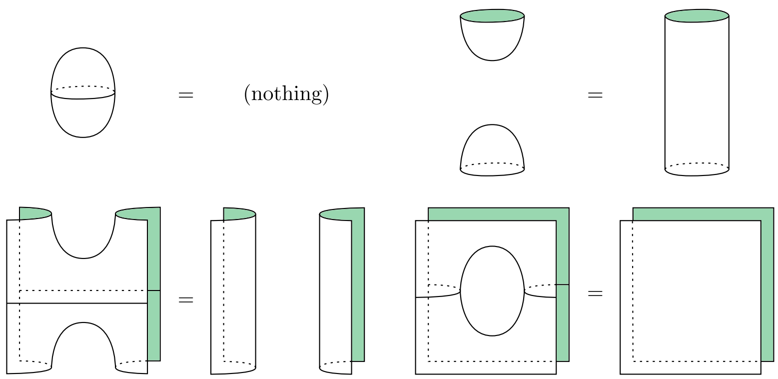
2-morphisms



and 3-morphisms

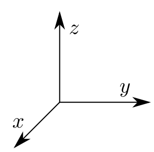


We require the 3-morphisms to satisfy invertibility relations:



Several remarks on the diagrams are in order.

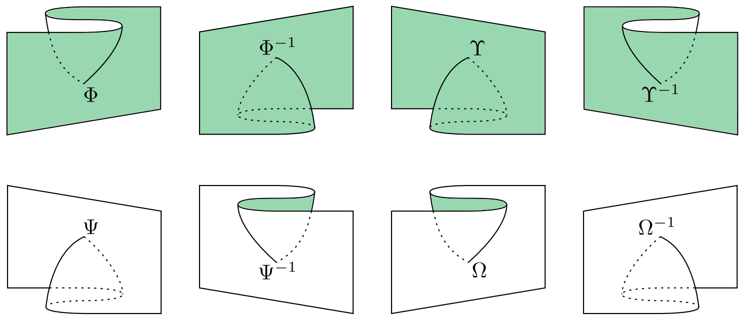
- In our surface diagrams, composition across 0-morphisms happens in the x direction, across 1-morphisms in the y direction, and across 2-morphisms in the z direction.



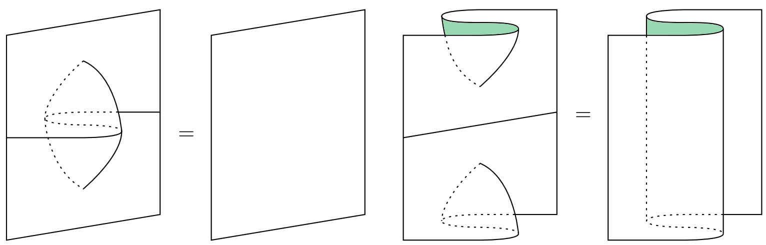
- We color one side of each surface. This is meant to be interpreted according to the following convention: the region on the non-colored side of a surface corresponds to the object X and the region on the colored side corresponds to Y.
- One could write the 2-morphisms on the fold lines of the surfaces. (Fold lines are boundaries of the projection which aren't boundary of the surface.) We have chosen not to do this, however, as it makes the diagrams busier and is not necessary given the coloring convention.
- As an example of our conventions, we note that the saddle Δ is a map from $1_{f \circ_0 g}$ to $\alpha \circ_1 \gamma$.
- We will need to reference various symmetries of the cube that act on the diagrams. R_x refers to reflection across the yz-plane, and similarly for R_y and R_z. Further, we'll refer to swapping the colored side of a surface as R_C. We allow ourselves to refer to R_C as a reflection for convenience.
- When we write a relation in the above definition, we actually mean to impose all relations of that form. To get the other relations, we act by all compositions of R_x, R_y, R_z and R_C.

Next, we define a variant of equivalence in which we demand that the morphisms at levels 0-2 form an up-to-isomorphism adjoint equivalence. This amounts to taking the coherence relations for a 2-categorical adjoint equivalence as data.

Definition. A *semi-coherent equivalence* in a Gray-category is an incoherent equivalence with extra 3-morphism data:

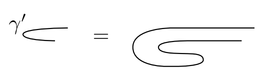


We require this extra data to satisfy invertibility relations:



Theorem (semi-coherification)**.** Any equivalence can be semi-coherified.

Proof. We define γ' and δ' in terms of α, β, γ, and δ as follows:



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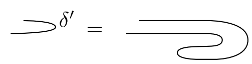
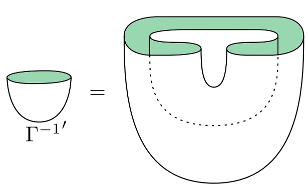


Proofs from the 2-categorical case now become data that we record. The cups, caps, and saddles for γ' and δ' can be defined as



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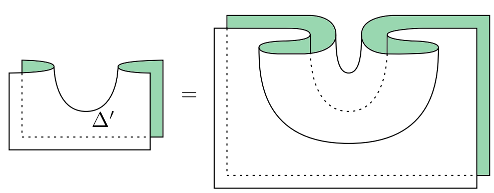
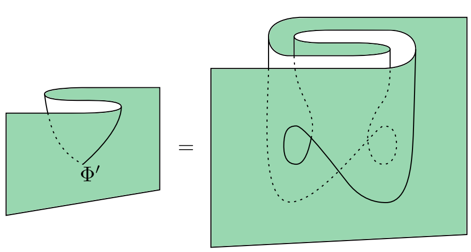


or various reflections thereof. We define cusps



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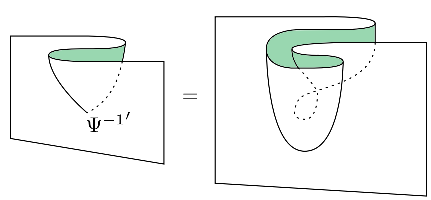
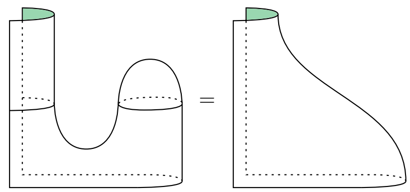


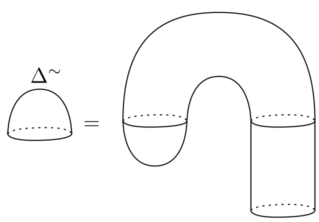
and get the other cusps as R_x, R_z, and $R_x R_z$ reflections of these. We automatically get the desired invertibility relations. Indeed, the R_z reflection of a diagram gives the inverse morphism. □

Morse-Cancellation

We will now consider a coherence relation on the 3-data: the Morse-cancellation.



This relation has a R_x symmetry, so there are $|\langle R_y, R_z, R_C\rangle| = 8$ Morse-cancellation relations. We can regard these relations as zig-zag relations. Indeed, in $\mathrm{Hom}(X, X)$, we have $\Gamma, \Gamma^{-1}, \Delta$, and Δ^{-1}, which form an incoherent 2-equivalence. This is straightforward to see: just flatten out the x-direction of a surface diagram to get a 2D-diagram. In other words, the fold lines of a surface diagram give 1-morphisms in the microcosm 2-category. According to the 2-categorical coherence argument, we can rechoose Γ and Γ^{-1} as



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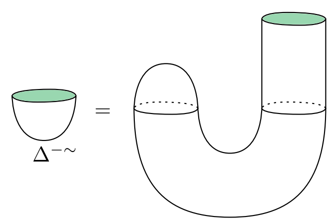
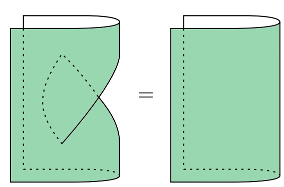


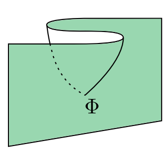
to give an adjoint equivalence with the Δ, Δ^{-1} saddles. One could equally well rechoose the saddles instead of the cup and cap. The reader may find it amusing to draw these, as well.

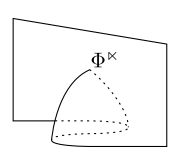
Swallowtail

We consider another coherence relation on 3-morphisms: the swallowtail.



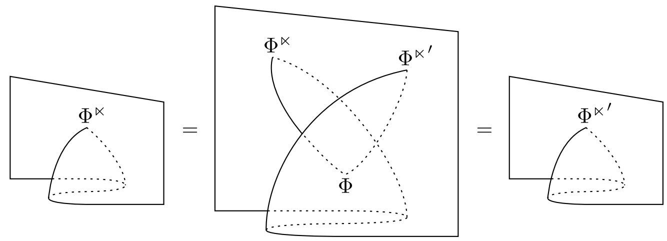
The swallowtail has a $R_x R_z$ symmetry (rotation about the y-axis) and so there are 8 variants. It can be thought of as a twisted zig-zag relation. Following this assertion, we will recreate the arguments for 2-categorical coherification, but in a twisted manner.

Definition. For any cusp, 

there is at most one cusp 

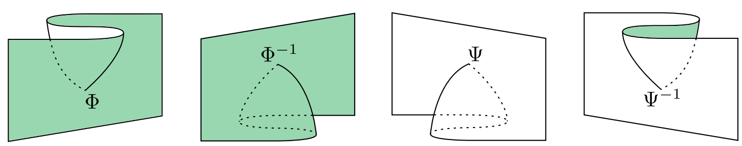
satisfying the swallowtail relations, which we'll call the *swallowtail partner*.

Proof. Consider making the right and left cancellations:



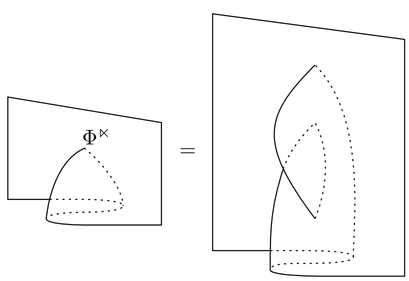
□

Lemma (existence of swallowtail partners)**.** Given cusps,



we may rechoose Ψ and Ψ^{-1} so that the cusps satisfy the swallowtail relations.

Proof. Let's define $\Phi^{\ltimes}$ and $\Phi^{-\ltimes}$ based on Φ, Φ^{-1}, Ψ, and Ψ^{-1} as follows:



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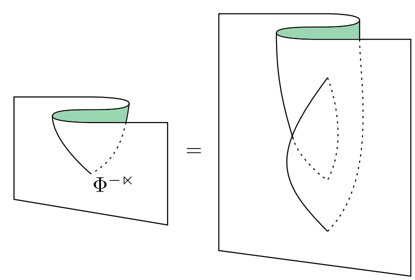
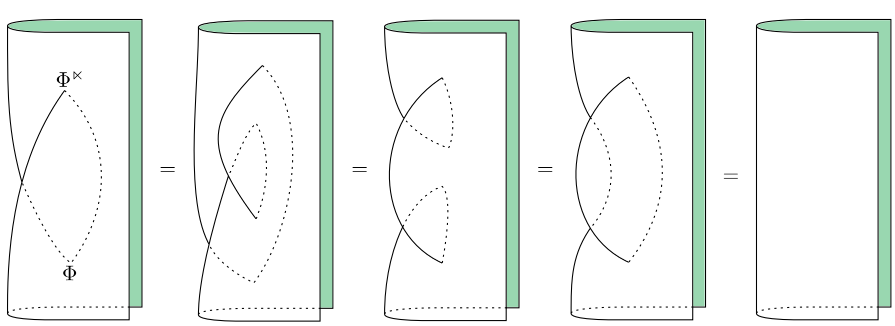
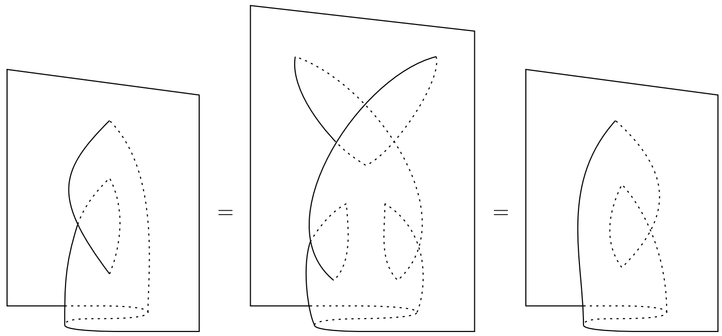


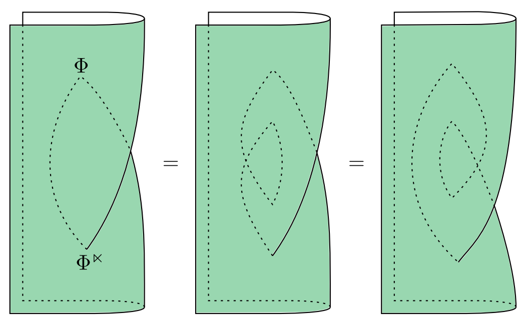
We now claim that Φ and $\Phi^{\ltimes}$ satisfy their two swallowtail relations, as do Φ^{-1} and $\Phi^{-\ltimes}$. We'll show the relations for Φ and $\Phi^{\ltimes}$ and just note that the relations for the inverses may be proved as R_z reflections. We get one swallowtail relation as follows:



Now consider the following



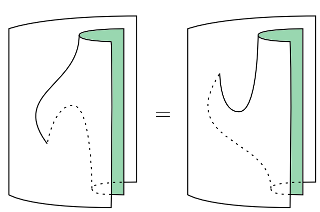
which lets us make the following reduction:



We can simplify the right-most diagram by using a $R_x R_y R_C$ reflection of the argument for the other swallowtail relation. □

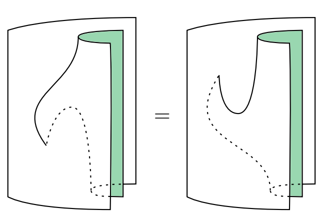
Cusp-Flip

In this section, we analyze the cusp-flip relations. We will assume that we have a semi-coherent equivalence with the Morse-cancellation and swallowtail relations at our disposal. The cusp-flip relation is given by the following diagram.

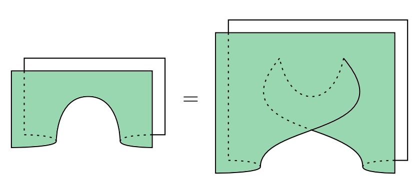


We can think of this rule as asserting that the the down and up deformations of a horizontal cusp must describe the same morphism. The cusp-flip relation has a $R_x R_z R_C$ symmetry, so there are 8 variants.

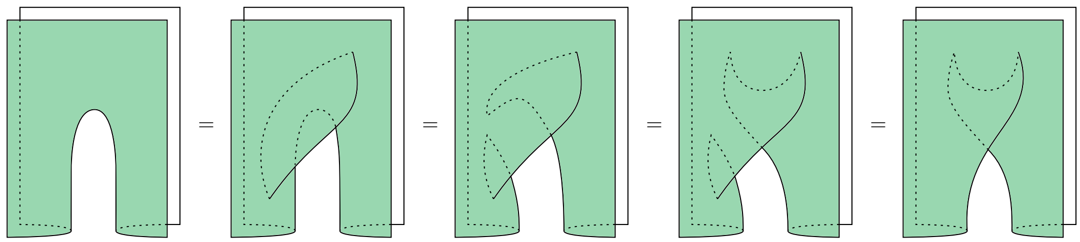
Lemma (solving for saddle)**.** The following relations imply each other:



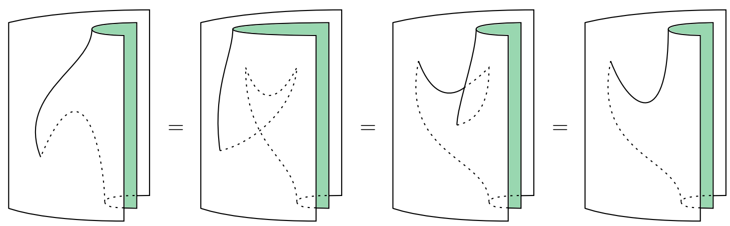
and



Proof. For the forward direction, consider

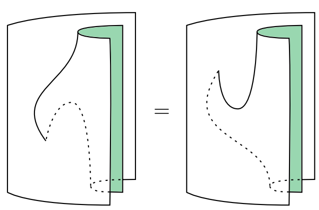


The reverse direction follows from

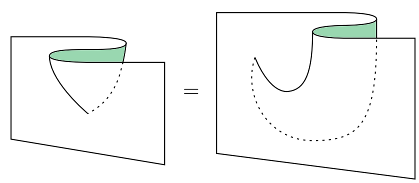


□

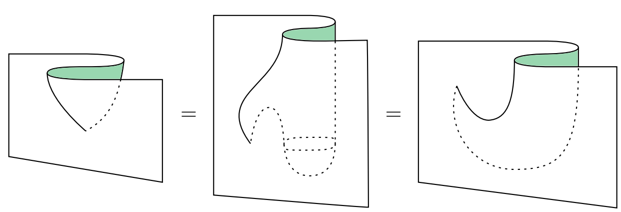
Lemma (solving for cusp)**.** The following relations imply each other:



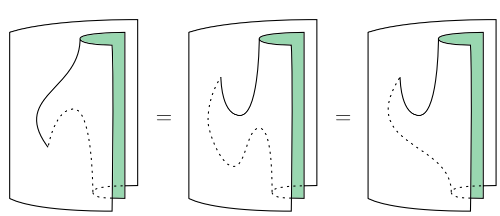
and



Proof. For the forward direction, consider



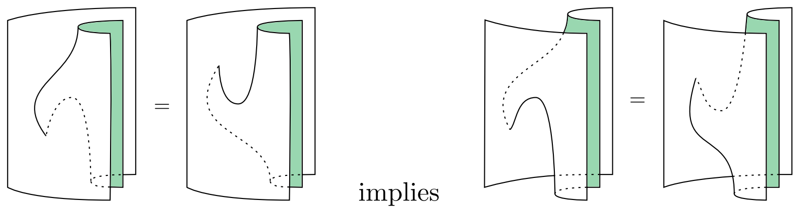
The reverse direction follows from



□

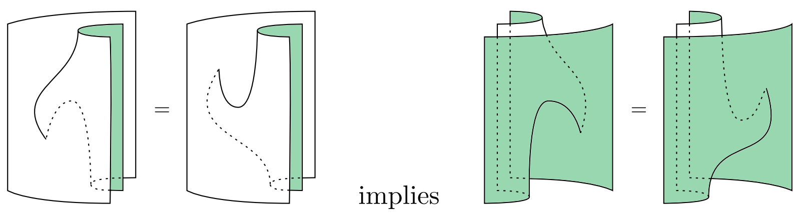
The last three lemmas of this section each say that a cusp-flip relation implies a reflected cusp-flip relation. Of course, each converse immediately holds, so we may regard each lemma as giving a logical equivalence.

Lemma (R_z)**.** A cusp-flip relation implies its R_z reflection cusp-flip.

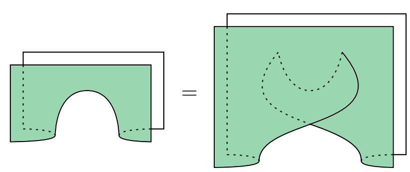


Proof. The data in the relation on the right are the inverses of the data in the relation on the left. The proof only involves vertical composition, so it's easy enough to read in algebraic notation: if $A = B$, then $A^{-1}AB^{-1} = A^{-1}BB^{-1}$ and so $B^{-1} = A^{-1}$. In other words, inverses are unique. □

Lemma (R_xR_y)**.** A cusp-flip relation implies its R_xR_y reflection cusp-flip.

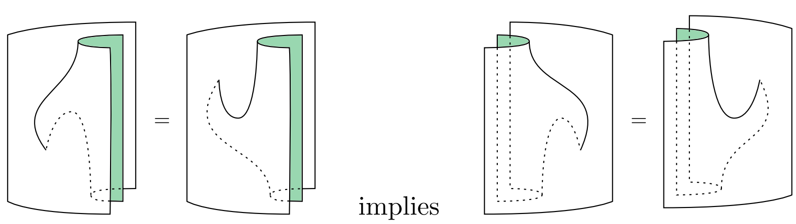


Proof. We showed in a previous lemma that the relation on the left is equivalent to

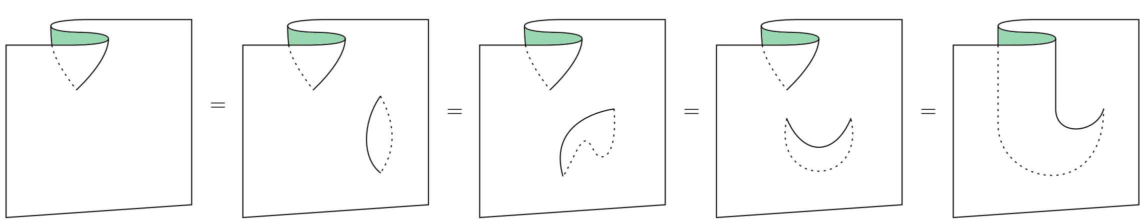


Further, an R_xR_y reflection of that same lemma shows that the relation on the right is also logically equivalent to this saddle relation. □

Lemma (R_y)**.** A cusp-flip relation implies its R_y reflection cusp-flip.



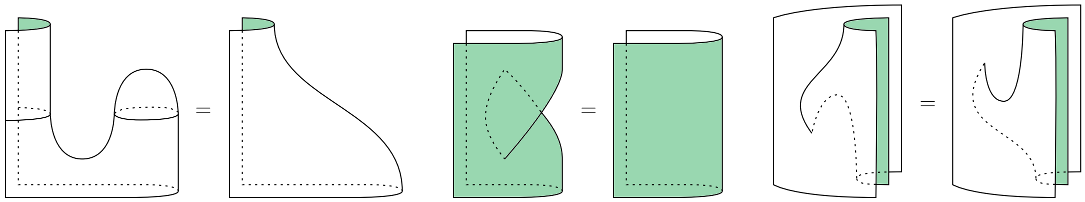
Proof. We'll prove the result using the alternate relations obtained by solving each for a cusp. Then



gives the result. □

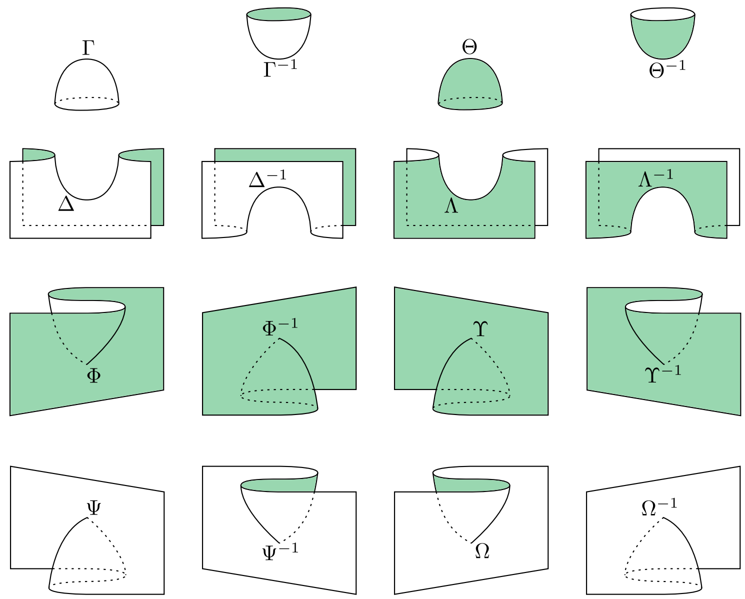
Coherification

Definition. A *fully coherent equivalence* in a Gray-category is a semi-coherent equivalence that further satisfies Morse-cancellations, swallowtails, and cusp-flips.

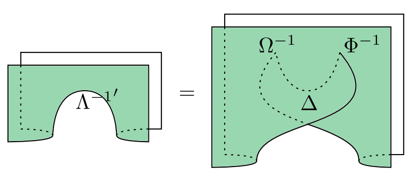


Theorem (coherification)**.** Any incoherent equivalence can be fully coherified.

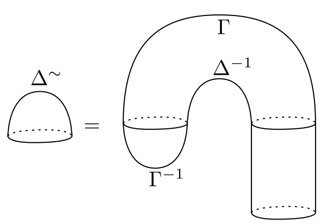
Proof. First, semi-coherify so that we have data:



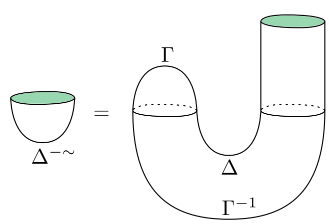
Rechoose saddle Λ^{-1} as $\Lambda^{-1'}$ in terms of Δ, Ω^{-1}, and Φ^{-1} as follows:



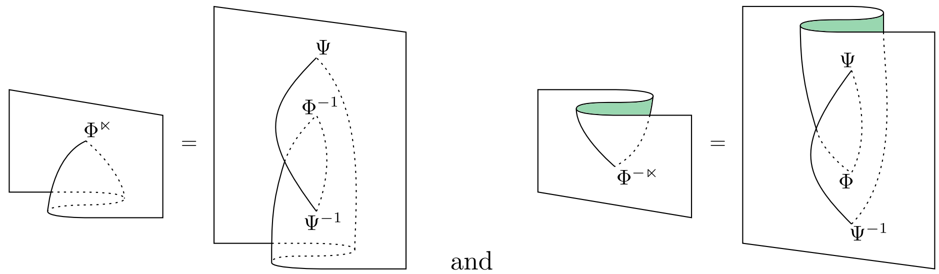
Similarly, rechoose Λ as Λ' in terms of Δ^{-1}, Ω, and Φ. These choices have ensured that at least one of the eight cusp-flip relations are satisfied. Next, rechoose the Γ and Γ^{-1} as $\Delta^{\sim}$ and $\Delta^{-\sim}$:



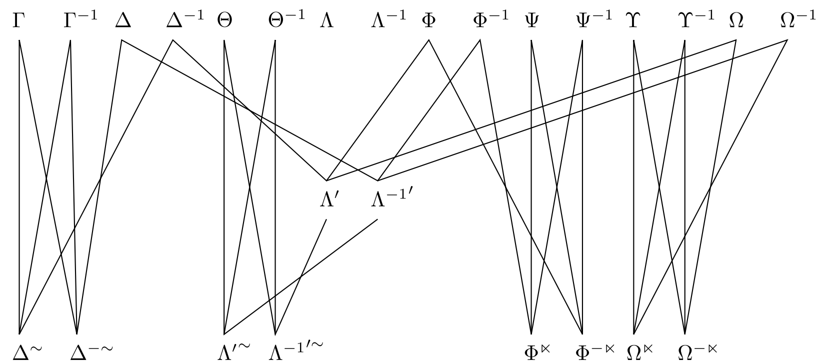
and



Similarly, rechoose Θ, and Θ^{-1} as $\Lambda'^{\sim}$ and $\Lambda^{-1'\sim}$. Rechoose Ψ and Ψ^{-1} as $\Phi^{\ltimes}$ and $\Phi^{-\ltimes}$:



Also, rechoose Υ and Υ^{-1} as $\Omega^{\ltimes}$ and $\Omega^{-\ltimes}$. The following tree summarizes our choices:



At the top, is the semi-coherified data. If we were feeling more adventurous, we could extend this tree back to the incoherent equivalence data. If we rechoose a piece of data, we place that directly below and note which other data went into that choice. We claim that the following data forms a fully coherent equivalence:

$$\Delta^{\sim}, \Delta^{-\sim}, \Delta, \Delta^{-1}, \Lambda'^{\sim}, \Lambda^{-1'^{\sim}}, \Lambda', \Lambda^{-1'}, \Phi, \Phi^{-1}, \Phi^{\ltimes}, \Phi^{-\ltimes}, \Omega^{\ltimes}, \Omega^{-\ltimes}, \Omega, \Omega^{-1}$$

We've ensured that all Morse-cancellations and swallowtails hold and one of the eight cusp-flip holds. By the lemmas of the cusp-flip section, all eight cusp-flip relations now hold, automatically. Indeed, R_z, R_xR_y, and R_y generate all cusp-flips given the $R_xR_zR_C$ symmetry of the cusp-flip relation. □

4 Future Work

I have a few questions that I'm still working on that may be added to a later draft of this paper.

How unique is coherification in a Gray-category?

The coherification result for equivalences in Gray-categories may be regarded as a statement of existence. That is, given any incoherent equivalence, we have give a procedure to build a coherent equivalence. I'm working on a uniqueness result: a coherent equivalence in a Gray-category should be determined by a small subset of its defining data.

What can we say about Tang_2^1*?*

The calculations in this paper should be regarded as taking place in a free Gray-category on the given generators, quotiented by the given invertibility and topological relations. There's another Gray-category that I'm interested in, Tang_2^1, which has all the same generating morphisms as a coherent equivalence, but is subject to only the topological relations. I've been looking at understanding maps between representations of Tang_2^1.

To what degree can we extend "mates" to surfaces?

Suppose a and b are parallel 1-morphisms in a 2-category with right adjoints. Then any map $\tau\colon a \to b$ has a corresponding mate $\tau^*\colon b^* \to a^*$. If τ happens to be an isomorphism, then a quick diagrammatic check shows the inverse of the mate is the mate of the inverse. The situation is more complicated with surfaces. Indeed, the existence of saddles requires us to be working with the ambidextrous adjunction. But then any τ induces $\mathbb{Z}$-many maps, half $a \to b$ and half $b^* \to a^*$. I'm working with a calculus of surfaces with seams to sort this all out.

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