MATH 20D NOTES

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Contents

1. About Differential Equations

These notes follow Professor Gross's class on Differential Equations and the eighth edition of *Elementary Differential Equations* by Boyce and DiPrima.

1.1. **The Setup.**

In Math 20D, we'll be trying to solve *ordinary differential equations* - ODEs. These are equations which involve derivatives, but not partial derivatives (PDEs). A *solution* to an ODE is a function which satisfies the equation. We only care that the solution be defined on some open interval. An ODE will typically have many solutions. We can ask for a particular solution by giving more information: an *initial condition.*

The *order* of a differential equation is the degree of the highest derivative that appears. A differential equation is *linear* if it is of the form

$$\frac{d^n y}{dt^n} + p_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_n y = g.$$

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A linear ODE is *homogeneous* if $g = 0$. Linear ODEs have well behaved solutions and there are techniques to solve them, so we'll spend much of our time thinking about them.

We typically picture a solution set to a differential equation by plotting y against t, and drawing short tangent lines to the solutions at many of the points (t, y). A particular solution is a curve which follows the "flow" of this field.

Notation. We'll give certain letters meanings:

- t**:** an independent variable.
- y**:** a dependent variable.
- x**:** a variable; may be dependent or independent depending on context.
- p, q, g**:** continuous functions of t.
- a, b, c**:** real number constants.

I won't usually display a function's arguments, unless I want to emphasize them - for example, I'll usually write p for $p(t)$.

1.2. **About Solutions.**

A solution y is *explicit* if we can write

$$y = (\text{stuff not involving } y).$$

An *implicit* solution is an equation defining y, in which we may not be able to solve for y.

There is a existence and uniqueness theorem that re-occurs throughout this course. It basically read as follows:

Theorem (Existence and Uniqueness)**.** Suppose we have an ODE that we want to solve for some initial values t_0. If the functions in the differential equation are reasonably behaved near the initial values, then there is a solution defined on some interval containing t_0. Further, this is the only solution satisfying these initial conditions.

In the case that the ODE is linear, we can say more: let I be the largest open interval containing t_0 for which the functions in the differential equation are well-behaved. Then, we get the solution defined on all of this I. Further, we can write a *general solution*:

$$c_1 y_1 + \cdots + c_n y_n + Y,$$

where the y_is are solutions to the homogeneous equation and Y is a particular solution to the nonhomogeneous equation.

Typical ways that functions in calculus misbehave:

(1) dividing by zero;
(2) taking the square root of a negative number;
(3) taking the log of non-positive number.

Theorem (Superposition)**.** Any linear combination of solutions to a homogeneous linear ODE is a solution.

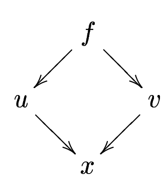
1.3. **Quick Calculus Review.**

Rule (Chain, single variable)**.** Suppose we have functions $f(u)$ and $u(x)$. Then

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}.$$

Rule (Chain, two variable)**.** Suppose we have functions $f(u, v)$, $u(x)$, and $v(x)$. Then

$$\frac{df}{dx} = \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx}.$$



Recall the difference in calculating a partial derivative and a derivative:

$$\frac{\partial}{\partial x}(x^2 y) = 2xy.$$
$$\frac{d}{dx}(x^2 y) = 2xy + x^2 \frac{dy}{dx}.$$

The integration by parts formula is

$$\int u\, dv = uv - \int v\, du.$$

We try to choose u so that it gets simpler upon differentiation, and dv so that it doesn't get much more complicated upon integration.

1.4. **Word Problems.** Interpreting the word problems can be a bit of a pain at first. I've found the following three things usually are a big help getting me through:

(1) Decide which variable will be differentiated.
(2) Put units on everything.
(3) Check whether positive/negative signs make sense.

2. First Order ODEs

2.1. **Linear.** A first order linear ODE is of the form

$$\frac{dy}{dt} + p(t)y = g(t).$$

To solve this, we define an integration factor:

$$\mu(t) = e^{\int p(t)\, dt}.$$

The function $\mu(t)$ is chosen so that, after multiplying our ODE through by $\mu(t)$, we can recognize the LHS as the derivative of $\mu(t)y$. Integrating both sides gives the solutions.

2.2. **Separable.** An ODE is separable if it can be written in the form

$$a(x) + b(y)\frac{dy}{dx} = 0.$$

Let $A(x)$ and $B(y)$ be anti-derivatives of $a(x)$ and $b(y)$ respectively. Then we can recognize the LHS as a chain rule:

$$\frac{d}{dx}(A(x) + B(y)) = 0.$$

Integrating both sides gives the solutions.

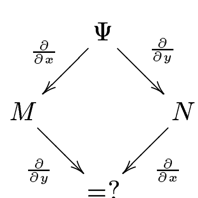
2.3. **Exact.** An ODE is exact if it is of the form

$$M(x,y) + N(x,y)\frac{\partial y}{\partial x} = 0,$$

where

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

We are using a theorem from multivariable calculus that says, over a simply connected domain, if we get equality on the lower half:



then Ψ must exist. To solve the exact equation, we recognize the LHS as $\frac{d\Psi}{dx}$ and integrate w.r.t x.

Sometimes we can use an integration factor to make an ODE exact, though this is difficult in general.

2.4. **Autonomous.** We'll think of our independent variable, t, as time. Then an autonomous equation is a rule which is time-invariant. That is, it is of the form

$$\frac{dy}{dt} = f(y).$$

An *equilibrium solution* is a constant solution, $y_e = K$. Such a K is a zero of $f(y)$. If solutions with initial values close to K approach K in the long run, we call y_e *stable*. If, on the other hand, solutions with initial values close to K diverge from K, we call y_e *unstable*.

3. Second Order Linear ODEs

3.1. **Homogeneous with Constant Coefficients.** Given,

$$ay'' + by' + cy = 0,$$

we find solutions according to the roots of the *characteristic equation*:

$$ar^2 + br + c = 0.$$

distinct real roots r_1, r_2: Take

$$y_1 = e^{r_1 t};$$
$$y_2 = e^{r_2 t}.$$

complex roots $\lambda \pm i\mu$: Take

$$y_1 = e^{\lambda t}\cos(\mu t);$$
$$y_2 = e^{\lambda t}\sin(\mu t).$$

repeated root $r = r_1 = r_2$: Take

$$y_1 = e^{rt};$$
$$y_2 = te^{rt}.$$

3.2. **The Wronskian.**

Definition. The *Wronskian* of solutions y_1 and y_2 is given by

$$W = y_1 y_2' - y_1' y_2.$$

Theorem (Spanning)**.** Suppose we have a homogeneous second order linear ODE. Suppose the Wronskian of the solutions y_1 and y_2 is non-zero for some point. Then any solution can be written as

$$y = c_1 y_1 + c_2 y_2.$$

Further, at any point where the Wronskian is non-zero, we can demand any initial condition.

Definition. Written with generic constants, we call $c_1 y_1 + c_2 y_2$ the *general solution* and say that y_1 and y_2 form a *fundamental set of solutions.*

Theorem. If f and g are differentiable on an open interval I and have Wronskian that is non-zero for some point in the interval, then f and g are linearly independent.

Remark. The converse of the previous is not true.

Theorem (Abel)**.** We can find the Wronskian, up to a scalar multiple, without knowing the solutions. Given,

$$y'' + py' + qy = 0,$$

then

$$W = ce^{-\int p(t)\,dt},$$

on an open interval for which p and q are continuous.

3.3. **Reduction of Order.** Suppose that we know one (non-trivial) solution y_1 of the second order homogeneous linear ODE, $y'' + py' + q = 0$. Suppose that we want to find a second solution. One can try the *reduction of order* technique:

Assume $y_2 = vy_1$, for some function v. As we want y_2 to be a solution of the ODE, we demand that it satisfy

$$y_2'' + py_2' + q = 0.$$

By calculating out some derivatives, one finds

$$y_1 v'' + (2y_1' + py_1)v' = 0.$$

Under the substitution, $z = v'$, this is a first order linear ODE in z, which we can solve with a separation of variables.

3.4. **Nonhomogeneous.** The *method of undetermined coefficients* is a technique for finding a particular solution to a nonhomogeneous linear equation with constant coefficients. Basically, you guess the form of a nonhomogeneous solution according to the RHS, try it out with variables for constants, and see if you can find numbers that make the constants work. See p. 181 of Boyce for hints about making a good guess.

Theorem (Variation of Parameters)**.** A specific solution of

$$y'' + py' + qy = g,$$

on an open interval I, where p, q, and g are continuous, is given by

$$Y = -y_1 \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(s)}\, ds + y_2 \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(s)}\, ds,$$

where t_0 is your favorite point in I.

4. Systems of First Order ODEs

In this section, we'll have one independent variable, t, and several dependent variables, $x_1, \ldots, x_n$. A system of first order ODEs is of the form:

$$\frac{dx_1}{dt} = F_1(t, x_1, \ldots, x_n),$$
$$\vdots$$
$$\frac{dx_n}{dt} = F_n(t, x_1, \ldots, x_n).$$

We can rewrite an nth order ODE as a system of n first order equations: introduce a variable x_j for the j-th derivative of y and throw in equations, $x_j' = x_{j+1}$.

A system of first order ODEs is *linear* if it is of the form

$$\frac{dx_1}{dt} = p_{11}x_1 + \cdots + p_{1n}x_n + g_1,$$
$$\vdots$$
$$\frac{dx_n}{dt} = p_{n1}x_1 + \cdots + p_{nn}x_n + g_n.$$

We can rewrite this as a matrix equation,

$$\frac{d}{dt}\mathbf{x} = \mathbf{P}\mathbf{x} + \mathbf{g}.$$

The existence and uniqueness theorem generalizes to systems of first order differential equations. Further, if the system is linear, then the solution's interval of definition may be chosen to be any interval where the p_{ij}s and g_ks are well-behaved.

4.1. **Homogeneous.** A first order linear *homogeneous* system of n ODEs may be written as a matrix equation:

$$\frac{d}{dt}\mathbf{x} = \mathbf{P}\mathbf{x}.$$

Theorem (Superposition)**.** Any linear combination of solutions to a homogeneous system is a solution.

Notation. We will index vectors with superscripts, so as not to confuse with components: $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$, etc. Note that this notation is not meant to indicate taking derivatives!

Definition. The matrix $\mathbf{X}$ has solutions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ as columns. The *Wronskian* is $\det(\mathbf{X})$.

Definition. A *fundamental set* of solutions is a basis for the solution space to a homogeneous linear system of ODEs.

Theorem. n linearly independent solutions to a homogeneous system form a fundamental set of solutions.

Theorem. On an interval where solutions exist, the Wronskian is either never zero or it is identically zero.

4.2. **Homogeneous with Constant Coefficients.** We will consider a system of n ODEs:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is an $n \times n$ matrix of real constants, with $\det(\mathbf{A}) \neq 0$.

If $\mathbf{A}$ has distinct real eigenvalues, $r_1, \dots, r_n$, and corresponding eigenvectors, $\xi^{(1)}, \dots, \xi^{(n)}$, then the system has a basis of solutions:

$$\begin{aligned}\mathbf{x}^{(1)} &= \xi^{(1)} e^{r_1 t},\\ &\vdots\\ \mathbf{x}^{(n)} &= \xi^{(n)} e^{r_n t}.\end{aligned}$$

Given a pair of complex conjugate eigenvalues, $\lambda + i\mu$ and $\lambda - i\mu$, with eigenvectors $\mathbf{a} + i\mathbf{b}$ and $\mathbf{a} - i\mathbf{b}$, we get real solutions

$$\begin{aligned}\mathbf{x}^{(1)} &= e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)),\\ \mathbf{x}^{(2)} &= e^{\lambda t}(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t)).\end{aligned}$$

If $\mathbf{A}$ has a repeated eigenvalue for which the geometric multiplicity equals the algebraic multiplicity, then proceed as before using the basis of eigenvectors. For a degenerate eigenvalue, ρ, we must work more: let ξ be an eigenvector for ρ. Then we get solutions,

$$\begin{aligned}\mathbf{x}^{(1)} &= \xi e^{\rho t},\\ \mathbf{x}^{(2)} &= \xi t e^{\rho t} + \eta e^{\rho t},\end{aligned}$$

where η satisfies

$$(\mathbf{A} - \rho\mathbf{I})\eta = \xi.$$

We call η a *generalized eigenvector* for ρ.

5. Power Series

5.1. **About Power Series.** We can solve more differential equations if we broaden our horizons a bit for what we accept as a solution. Power series are well-behaved and locally-defined functions. Many differential equations have solutions that can be expressed as power series. And by truncating a series, we still get a good local approximation. So it's worthwhile to spend some time working with these series.

A *power series* in x about x_0 is a series of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

Two power series are equal if and only if their coefficients agree.

Theorem (Ratio Test)**.** A power series converges for at any x for which the ratio of successive terms is strictly less than one. It diverges at any x for which the ratio of successive terms is strictly greater than one.

A point x_0 is *analytic* if there is some radius ρ about x_0 for which the power series converges at every point within ρ of x_0. Power series can be added, subtracted, and differentiated term-wise. They can be multiplied as generalized polynomials. The nth coefficient can be found by repeated differentiation:

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

5.2. **Solving Second Order Linear ODEs.** One can sometimes solve second order homogeneous linear ODEs by assuming a power series solution and looking for recurrence relations amongst the coefficients. One then expects that all coefficients may be written in terms of a_0 and a_1. A good example is the ODE defining sin and cos:

$$\frac{d^2y}{dx^2} + y = 0.$$

The even terms of the power series solution give cos and the odd terms give sin.

Theorem. Consider the differential equation:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0.$$

Let ρ_p be the radius of convergence for the power series for p and similarity for ρ_q. There exist two linearly independent solutions as power series and their radii of convergence are at least $\min(\rho_p, \rho_q)$.

If p is a polynomial, then one can find its radius of convergence by finding the distance to the roots of its denominator.

6. The Laplace Transform

We define the Laplace transform $\mathcal{L}$ by

$$\mathcal{L}[f] = \int_0^\infty e^{-st} f(t)\, dt.$$

We often write the resulting function as $F(s)$. The Laplace transform is *linear*:

$$\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g].$$

Theorem. Suppose f is a piecewise continuous function whose long-term growth is at most e^{at}. Then the Laplace transform exists for $s > a$.

Theorem. The Laplace transform has an inverse, L^{-1}, which is an invertible linear operator.

Theorem. Under suitable hypotheses,

$$\mathcal{L}[f^{(n)}] = s^n \mathcal{L}[f] - s^{n-1} f(0) - \cdots - f^{(n-1)}(0).$$

The Laplace transform turns derivatives into degree n polynomials in s. So we can transform a differential equation into an algebraic equation, solve the algebraic equation, and then untransform the solution.

$f(t)$	$F(s) = \mathcal{L}[f(t)]$
e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$

See p. 319 of Boyce for a longer table which translates between functions and their Laplace transforms.

Definition. The *Heaviside function* u_c is given by

$$u_c(t) = \begin{cases} 0, & \text{if } t < c; \\ 1, & \text{if } t \geq c. \end{cases}$$

We define two more linear operators: translation and phase-shift.

Definition. Suppose f is a piecewise-continuous function. The *translation* of f by a constant c, written $\mathcal{T}_c[f]$, is the piecewise-continuous function given by:

$$\mathcal{T}_c[f](t) = u_c(t) f(t - c).$$

Definition. The *phase-shift* of a function f by a constant c, written $\mathcal{P}_c[f]$, is given by:

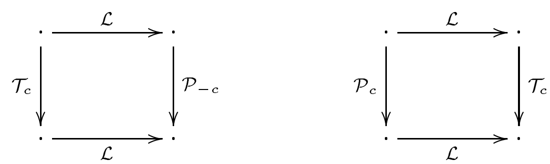
$$\mathcal{P}_c[f](t) = e^{ct} f(t).$$

Theorem. For reasonable f,

$$\mathcal{L}[\mathcal{T}_c[f]] = \mathcal{P}_{-c}[\mathcal{L}[f]]$$

holds on any interval (a, ∞) on which the transform exists. Similarly,

$$\mathcal{L}[\mathcal{P}_c[f]] = \mathcal{T}_c[\mathcal{L}[f]].$$



Definition. The *Dirac delta function* is a generalized function with the property

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0),$$

for any continuous f.

We write $\mathcal{L}[\delta] = 1$, though, we're not going into generalized-function theory.

Remark (Ignore this)**.** This result is consistent with saying that $\mathcal{L}$ is map of algebras

$$(\{\text{gen. functions}\}, \text{convolution}) \longrightarrow (\{\text{gen. functions}\}, \text{pointwise multiplication}).$$