

## CLOSURE AND INTERIOR

**Definition.** Given two sets  $C$  and  $D$ , I'll say  $C$  is *smaller than*  $D$  if  $C \subset D$ . I'll say  $C$  is *larger than*  $D$  if  $C \supset D$ .

**Warning.** Given two random sets, typically neither will be larger.

**Convention.** In today's lecture, we will take  $X$  to be a topological space and  $A \subset X$  a subset.

**Warning.** Although we could think of  $A$  as a *subspace* by giving it the subspace topology, we won't! That is, "closed" will always refer to  $X$ 's topology.

**Definition.** The *closure* of  $A$  (in  $X$ ) is given as follows:

$$\bar{A} = \bigcap_{\substack{K \supset A \\ K \text{ closed}}} K.$$

**Remark.** Notice that the intersection is closed. Also, given any  $K$  closed set containing  $A$ , the intersection is smaller. This gives us an alternate definition for "closure," with a nicer ring to it:

**Definition (Alternate).** The *closure* of  $A$  is the smallest closed set containing  $A$ .

**Definition.** The *interior* of  $A$  (in  $X$ ) is the largest open set sitting inside  $A$ . That is,

$$\text{Int}(A) = \bigcup_{\substack{U \subset A \\ U \text{ open}}} U.$$

**Remark.** It is clear that

$$\text{Int}(A) \subset A \subset \bar{A}.$$

**Example.** Consider  $A = (0, 1] \cup \{2\} \subset \mathbb{R}$ . Then

$$\begin{aligned} \bar{A} &= [0, 1] \cup \{2\}; \\ \text{Int}(A) &= (0, 1). \end{aligned}$$

**Lemma.** Let  $A \subset X$ . Then

- (1)  $A = \bar{A} \iff A$  is closed.
- (2)  $A = \text{Int}(A) \iff A$  is open.

*Proof.* We'll prove the statement for interior - closure is similar.

$\Rightarrow$ :  $A = \text{Int}(A)$  is open.

$\Leftarrow$ : Suppose  $A$  is open.  $\text{Int}(A)$  is the largest open set sitting inside  $A$ . But  $A$  itself is open and there aren't any sets sitting inside  $A$  larger than  $A$  itself. □

**Lemma.** Suppose  $A \subset B$ . Then

- (1)  $\bar{A} \subset \bar{B}$  and
- (2)  $\text{Int}(A) \subset \text{Int}(B)$ .

*Proof.*  $\text{Int}(A) \subset A \subset B$  and  $\text{Int}(A)$  is open. By definition,  $\text{Int}(B)$  is the largest open set sitting in  $B$ , so  $\text{Int}(B)$  is larger than  $\text{Int}(A)$ .

The statement for closures is part of the homework assignment. □

**Lemma.** Let  $A \subset X$ . Then

- (1)  $\overline{\bar{A}} = \bar{A}$ , and
- (2)  $\text{Int}(\text{Int}(A)) = \text{Int}(A)$ .

*Proof.*  $\overline{\bar{A}}$  is the smallest closed set containing  $\bar{A}$ . But  $\bar{A}$  is itself closed and there can't be a smaller set containing  $\bar{A}$ , so it fits the bill. The argument for interior is similar. □

**Question.** Can you think of a set for which the interior is equal to the closure?

**Answer.** Any set which is both open in closed. So for example, in any space,  $\emptyset$  and the whole space work.

**Lemma.** Let  $A, B \subset X$ . Then

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

*Proof.* Part of the homework assignment. □

**Remark.** The same does *not* hold for intersection: in  $\mathbb{R}$ ,

$$\overline{(-1, 0) \cap (0, 1)} = \overline{\emptyset} = \emptyset, \text{ but } \overline{(-1, 0)} \cap \overline{(0, 1)} = [-1, 0] \cap [0, 1] = \{0\}.$$

**Question.** Would you expect  $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$ ? Or  $\text{Int}(A \cup B) = \text{Int}(A) \cup \text{Int}(B)$ ?

**Definition.** Let  $x \in X$  be a point of a topological space. A *neighborhood* of  $x$  is an open set containing  $x$ .

**Theorem.** The following are equivalent:

- (1)  $x \in \overline{A}$ .
- (2) every neighborhood of  $x$  intersects  $A$ .
- (3) every basis element containing  $x$  intersects  $A$ .

*Proof.* We'll show  $1 \iff 2$  and  $2 \iff 3$ .

- 1  $\implies$  2: By contrapositive. Suppose there is a neighborhood  $U \ni x$  not intersecting  $A$ . Then  $x \notin U^c$ , a closed set containing  $A$ . Notice  $U^c \supset \overline{A}$ , hence  $x \notin \overline{A}$ .
- 2  $\implies$  1: By contrapositive. Suppose  $x \notin \overline{A}$ . Then take  $U = (\overline{A})^c$ .
- 2  $\implies$  3: Nothing to do.
- 3  $\implies$  2: Suppose  $U \ni x$  is a neighborhood. Then there is a basis element  $B$  satisfying  $U \supset B \ni x$ . But  $B$  must intersect  $A$ , hence, so does  $U$ .

□

**Definition.** Suppose  $A \subset X$  is a subset. A point  $x \in X$  is a *limit point* if every neighborhood of  $x$  intersects  $A$  at some point other than  $x$ . We write  $A'$  for the set of limit points.

**Warning.**  $A'$  may contain some points that are in  $A$  and some other points that are not in  $A$ . Consider

$$A = (0, 1] \cup \{2\} \subset \mathbb{R}.$$

One can show that  $A' = [0, 1]$ . Note  $0 \in A' \setminus A$  and  $2 \in A \setminus A'$ .

**Theorem.**  $\overline{A} = A \cup A'$ .

*Proof.*

- C: Let  $x \in \overline{A}$ . If  $x \in A$ , then we're done, so assume  $x \notin A$ . By the previous theorem, every neighborhood of  $x$  intersects  $A$ . But this is not at  $x$ , for  $x \notin A$ . Thus  $x \in A'$ .
- D:  $A \subset \overline{A}$ , so take  $x \in A'$ . Every neighborhood of  $x$  intersects  $A$ , so by the previous theorem,  $x \in \overline{A}$ .

□

**Corollary.** A subset is closed if and only if it contains its limit points.

If time permits, show a Hasse diagram for a 3 point space and how we take closures, interiors.