Talk on A_∞-Algebras

This talk is based on Bernhard Keller's papers on A_∞-algebras and *Operads in Algebra Topology and Physics*. It mostly follows Keller's *An Introduction to A-Infinity algebras and Modules*. I hope to give enough of the details to give flavor, but not so many as to be overwhelming.

Definitions

Definition. A linear map $f\colon M_\bullet \to N_\bullet$ between chain complexes is *homogeneous of degree n* if it sends elements of M_i to N_{i+n}.

Definition. Let $\mathrm{Hom}_k(M_\bullet, N_\bullet)$ denote the set of homogenous maps. It is graded by degree and we give it the differential

$$d(f) = d_N \circ f - (-1)^{|f|} f \circ d_M.$$

Remark. A *chain map* is a degree 0 map which is a cycle under this differential.

Definition. A *dg algebra* is the following data:

- a graded vector space $A = \bigoplus_{i\in\mathbb{Z}} A_i$,
- $d\colon A \to A$ of degree 1, and
- $m\colon A \otimes A \to A$ of degree 0.

with the following relations:

$$\begin{cases} d \circ d = 0, & d \text{ is a differential;} \\ d \circ m = m \circ (d \otimes 1 + 1 \otimes d), & m \text{ is a chain map;} \\ m \circ (m \otimes 1) = m \circ (1 \otimes m), & m \text{ is associative.} \end{cases}$$

Example. $\mathrm{Hom}_k(M_\bullet, M_\bullet)$ is a dg-algebra with composition.

Remark. If we instead look at $\mathrm{Hom}_k(M_\bullet, N_\bullet)$, we get the structure of a *dg-category*.

Definition. An A_∞*-algebra* is the following data:

- a graded vector space $A = \bigoplus_{i\in\mathbb{Z}} A_i$,
- $m_1\colon A \to A$ of degree 1,
- $m_2\colon A \otimes A \to A$ of degree 0,

  $\vdots$
- $m_n\colon A^{\otimes n} \to A$ of degree $2-n$,

  $\vdots$

with the following relations:

$$\begin{cases} m_1 \circ m_1 = 0, & m_1 \text{ is a differential;} \\ m_1 \circ m_2 - m_2 \circ (m_1 \otimes 1 + 1 \otimes m_1) = 0, & m_2 \text{ is a chain map;} \\ \vdots & \\ \sum_{\substack{n=r+s+t \\ r,t\geq 0 \\ s\geq 1}} (-1)^{r+st} m_{r+1+t} \circ (1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0. & \\ \vdots & \end{cases}$$

Remark. Notice that we've dropped the requirement that m_2 be associative. Instead, we have a rule

$$m_2 \circ (1 \otimes m_2 - m_2 \otimes 1) = m_1 \circ m_3 + m_3 \circ (m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1).$$

So m_2 is associative up to a boundary of m_3. In particular, m_2 induces an associative multiplication on H^*A.

Remark. We could go back and redefine a dg algebra as an A_∞-algebra with trivial m_ns for $n \geq 3$.

Definition. A *morphism* $f\colon A \to B$ of A_∞-algebras is a sequence of maps

$$f_n\colon A^{\otimes n} \to B$$

degree $1-n$ maps satisfying many relations amongst all the m_ns. In particular, we require that f_1 be a chain map and that

$$f_1 \circ m_2 - m_2 \circ (f_1 \otimes f_1) = m_1 \circ f_2 + f_2 \circ (m_1 \otimes 1 + 1 \otimes m_1);$$

that is, f_1 commutes with m_2 up to a chain homotopy f_2.

Remark. Composition of morphisms is basically as expected:

$$(f \circ g)_n = \sum_{i_1+\cdots+i_s=n} f_s \circ (g_{i_1} \otimes \cdots \otimes g_{i_s}).$$

Definition. A morphism is *strict* if $f_i = 0$ for all $i \neq 1$.

Definition. A morphism f is a *quasi-isomorphism* if f_1 induces an isomorphism on homology.

Remark. One can also define a homotopy of morphisms of A_∞-algebras.

Theorem (Prouté)**.** .

(1) Homotopy is an equivalence relation on $\mathrm{Hom}_{A_\infty\text{-alg}}(A, B)$.
(2) A morphism of A_∞-algebras is a quasi-isomorphism if and only if it is a homotopy equivalence.

Remark. The second part of the previous theorem is cool. We usually expect homotopy equivalences to be quasi-isomorphisms, but not vice-versa.

A_∞-Spaces

We want to look at spaces which have a binary multiplication which is "almost associative" in a continuously balanced sort of way.

Definition. The nth *associahedra* K_n is a $n-2$ dimensional polytope which parameterizes the ways to reassociate an n-fold multiplication.

Example. The first few associahedra:

- $K_2 =$ point;
- $K_3 = [0, 1]$;
- $K_4 =$ pentagon;
- $K_5 =$ cool diagram.

Definition. An *A_∞-space* is a topological space X with maps

$$\mu_n\colon K_n \times X^n \to X, \quad n \geq 2$$

satisfying suitable conditions.

Remark. μ_2 is really the multiplication we care about - higher multiplications are there just to sort out reassociations.

Example. Let Y be a pointed topological space. Then

$$\Omega Y = \mathrm{Hom}(S^1, Y)$$

with its usual topology is an A_∞-space.

Remark. The multiplication is non-associative - see picture of a line with breaks. But it's clear that homotopies exist to reassociate.

FIX ME. *How does the singular chain complex of an A_∞-space have an A_∞-algebra structure?*

Theorem. A connected space Y is homotopy equivalent to a based loop space if and only if Y admits the structure of an A_∞-space.

Remark. We require in the previous that Y be homotopy equivalent to a based CW-complex.

The Bar Construction

Now we'll look at an alternate way to characterize an A_∞-algebra.

Definition. Let V be a graded vector space. Then

$$\overline{T}V = V \oplus V^{\otimes 2} \oplus \cdots$$

is the reduced tensor algebra. We make $\overline{T}V$ a graded coalgebra by the comultiplication

$$\Delta\colon \overline{T}V \to \overline{T}V \otimes \overline{T}V$$

defined by

$$\Delta(v_1, \ldots, v_n) = \sum_{1 \le i \le n} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n).$$

Definition. A *coderivation* $b\colon \overline{T}V \to \overline{T}V$ is a map satisfying

$$\Delta b = (b \otimes 1 + 1 \otimes b)\Delta.$$

Lemma. Maps $\overline{T}V \to V$ are in bijection with coderivations $\overline{T}V \to \overline{T}V$.

Suppose A is an A_∞-algebra with multiplication maps. So we have

$$\begin{aligned} &m_n\colon A^{\otimes n} \to A; && \\ &b_n\colon (SA)^{\otimes n} \to SA, && \text{by suspending.} \\ &b\colon \overline{T}SA \to SA, && \text{by collecting up.} \\ &b\colon \overline{T}SA \to \overline{T}SA, && \text{by a univ. property(!)} \end{aligned}$$

Theorem. The previous association $\{m_n\} \mapsto b$ is a bijection. Further, a graded vector space with multiplications $\{m_n\}$ is an A_∞-algebra if and only if the coderivation b satisfies $b^2 = 0$.

Minimal Models

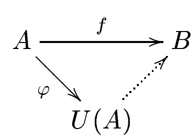
Theorem (Kadeishvili)**.** Let A be an A_∞-algebra. The homology H^*A has an A_∞-algebra structure such that

(1) $m_1 = 0$,
(2) m_2 is induced by that from A
(3) $1\colon H^*A \to H^*A$ lifts to a quasi-isomorphism of A_∞-algebras $H^*A \to A$.

Moreover, this structure is unique up to A_∞-isomorphism.

Definition. An A_∞-algebra A is *minimal* if it has $m_0 = 0$. We call H^*A the *minimal model* for A. A is *formal* if the minimal model can be chosen with m_n vanishing for all $n \neq 2$.

Theorem (anti-minimal model)**.** There is a universal A_∞-algebra morphism $\varphi\colon A \to U(A)$ to a dg algebra: for each dg algebra B and morphism of A_∞-algebras $f\colon A \to B$ there is a unique map of dg algebras making the factorization:



Moreover, φ is an A_∞-quasi-isomorphism.

Remark. The previous two theorems tell us that, up to A_∞-quasi-isomorphism, any A_∞ algebra can be taken to have $m_1 = 0$ and $m_n, n \geq 3$ non-trivial, or can be taken to have m_1 non-trivial and $m_n = 0, n \geq 3$.

Remark. Any dg algebra is an A_∞-algebra. And any A_∞-algebra is A_∞-quasi-isomorphic to a dg algebra. So up to A_∞-quasi-iso, there are as many dg algebras as A_∞-algebras. So what we're really gaining with A_∞-algebras is the minimal model description.

FIX ME. *How does this compare to (non A_∞) quasi-isomorphism classes of dg algebras?*

Derived Category

Fix an associative k-algebra A with unit. To get the derived category, one usually goes through a few categories:

- $\text{Mod}-A$**:** A-modules
- $\mathcal{C}A$**:** chain complexes of A-modules
- $\mathcal{H}A$**:** the homotopy category
- $\mathcal{D}A$**:** the derived category

One takes chain complexes, inverts nullhomotopic maps, and then inverts the quasi-isomorphisms. The process is similar over an A_∞-algebra.

Definition. An A_∞-module M over A is a $\mathbb{Z}$-graded vector space with multiplications

- $m_1 \colon M \to M$, degree 1 differential;
- $m_2 \colon M \otimes A \to M$, degree 0 action map;
- $m_3 \colon M \otimes A^{\otimes 2} \to M$, degree -1;
- $\vdots$

with plenty of relations.

Remark. An A_∞-module is comparable to a chain complex of modules.

Notation. The category of A_∞-modules over an A_∞-algebra A is denoted $\mathcal{C}_\infty A$.

Remark. One should also define morphisms and homotopies of A_∞-modules.

Definition. The *homotopy category* $\mathcal{H}_\infty A$ has the same objects as $\mathcal{C}_\infty A$ and we invert all nullhomotopies.

Remark. As it turns out, quasi-isomorphism of A_∞-modules are homotopy equivalences.

Definition. $\mathcal{D}_\infty A = \mathcal{H}_\infty A$.

Definition. For any ordinary associative algebra with unit A, there is a canonical functor

$$\mathcal{I} \colon \mathcal{D}A \to \mathcal{D}_\infty A.$$

Theorem. .

- $\mathcal{I}$ is an equivalence onto a full subcategory of $\mathcal{D}_\infty A$ which is closed under taking homology.
- Let M be an A_∞-module M. Then H^*M has an A_∞-module structure with $m_1 = 0$. Further $M \cong H^*M$ in $\mathcal{D}_\infty A$.

Remark. By keeping around A_∞ structure, we can recognize a module (up to quasi-iso) from its homology. Indeed, by the full-faithfulness of $\mathcal{I}$,

$$\begin{aligned} M_1 \cong M_2 & \qquad \text{in } \mathcal{D}A \\ \iff & \\ M_1 \cong M_2 & \qquad \text{in } \mathcal{D}_\infty A \\ \iff & \\ H^*M_1 \cong H^*M_2 & \qquad \text{in } \mathcal{D}_\infty A. \end{aligned}$$