An esoteric subject noted for its difficulty and irrelevance

The title is a quote of Moore and Seiberg in *Classical and quantum conformal field theory* in reference to category theory.

Long ago, when shepherds wanted to see if two herds of sheep were isomorphic, they would look for an explicit isomorphism. In other words, they would line up both herds and try to match each sheep in one herd with a sheep in the other. But one day, along came a shepherd who invented decategorification. She realized one could take each herd and "count" it, setting up an isomorphism between it and some set of "numbers," which were nonsense words like "one, two, three, ..." specially designed for this purpose. By comparing the resulting numbers, she could show that two herds were isomorphic without explicitly establishing an isomorphism!

John Baez

1. Finite Sets

Any equality is inherently boring - we are always saying that something is itself. It is much more interesting to say how to objects are the same - that is, to give an isomorphism. Indeed, it is central to Galois theory that we study the automorphisms of an object.

Let's start math over again, from the natural numbers, but doing it right: let's keep around isomorphisms. Instead of working with natural numbers, let work with finite sets. And let's replace equalities of numbers with a choice of bijection. It's worth mentioning that combinatorialists do this sort of thing all the time.

We all know how to multiply two sets A and B.

$$A \times B = \{(a, b) \colon a \in A; b \in B\}.$$

To verify that we're staying consistent with the natural numbers, we take cardinalities:

$$|A \times B| = |A||B|.$$

We also like adding numbers. So how do we add finite sets? The disjoint union does the trick:

$$A + B = \{(a, \text{red})\} \cup \{(b, \text{blue})\}.$$

Of course my choice of "painting" red and blue wasn't particularly important - but it was important that I chose some way to make the sets disjoint. And if we had a different paint job, that would be ok, because it would be obvious how to "translate" between the two.

Our definition is justified:

$$|A + B| = |A| + |B|.$$

Before going any further, we should make sure that we have identities for each of our operations. For addition, we can use the empty set. Then for any set A, there are bijections

$$A + \varnothing \longleftrightarrow A \longleftrightarrow \varnothing + A.$$

Similarly, there are bijections

$$A \times \{73\} \longleftrightarrow A \longleftrightarrow \{73\} \times A.$$

Again, the choice of 73 for our singleton set isn't important - just that we have a singleton set.

We can also check that our operations are associative, commutative, and distribute, as long as we aren't picky about having equalities. These proofs are very straightforward and visual:

(picture of commutativity proof for multiplication)

1.1. Exponentiation.

Now, it's time for the most important definition of this talk: for sets A and B, we define

$$B^A = \{\text{functions } A \to B\}.$$

As usual, we can check that this is reasonable:

$$|B^A| = |B|^{|A|}.$$

We have to be a little careful with the empty set. Given any range set B, there is exactly one function $\varnothing \to B$ - the empty function.

Actually, we have defined enough now that we can state a few familiar identities:

$$(B \times C)^A \longleftrightarrow B^A \times C^A \tag{1}$$

$$A^{B+C} \longleftrightarrow A^B \times A^C \tag{2}$$

$$C^{B \times A} \longleftrightarrow (C^B)^A \tag{3}$$

As we're all too well aware, in modern mathematics, we don't always work in the realm of finite sets. We'll work with vector spaces, groups, topological spaces, etc. We define B^A in each of these contexts to be the set of functions which are appropriate to the realm: linear maps, homomorphisms, continuous maps, etc.

I want to stress something here: if A and B are vector spaces, we take

$$B^A = \text{ the } \textit{set} \text{ of linear maps } A \to B.$$

It's also true that B^A can be thought of as a vector space itself - this is called enrichment - but in many contexts we can't do it, so we don't assume it.

With this in mind, let's take second look at equation one above. Suppose A, B, and C are groups. On the right hand side, we have the product of sets of homomorphisms. On the left hand side, we have $B \times C$ - this needs to be a group, not just a set. The direct product of groups does the trick nicely. In fact, in we weren't very familiar with group theory, we could take equation 1 to be a *definition* of the product of groups! Equation two gives us a definition for the sum of two groups. It's usually known as the "free product" of groups. And these equations work to define the usual "product" and "coproduct" in other realms.

Theorem. Fix B and C. Any two products of B and C are canonically isomorphic.

Proof. Suppose P and Q are both products of B and C. Then we have fixed bijections

$$P^A \longleftrightarrow B^A \times C^A \longleftrightarrow Q^A.$$

Taking $A = P$, we have $1_P \in P^P$ and we can follow through the bijections giving $\pi_B \in B^P$, $\pi_C \in C^P$, and $f \in Q^P$. Similarly, we can take $A = Q$ and follow $1_Q \in Q^Q$ through the bijections giving $\rho_B \in B^Q$, $\rho_C \in C^Q$, and $g \in P^Q$. Consider $g \circ f \in P^P$:

$$\pi_B \circ g \circ f = \rho_B \circ 1_Q \circ f = \rho_B \circ f = \pi_B$$

and similarly

$$\pi_C \circ g \circ f = \pi_C.$$

Note, under our bijections

$$\begin{array}{ccc} 1 & \longleftrightarrow & (\pi_B, \pi_C) \\ & & \| \\ g \circ f & \longleftrightarrow & (\pi_B \circ g \circ f, \pi_C \circ g \circ f) \end{array}$$

As $g \circ f$ and 1_P go to the same place, $g \circ f = 1_P$. Similarly $f \circ g = 1_Q$. □

We've done something cool - we're able to define a "product" in many contexts by declaring what we expect of its functions. And while the definition doesn't give a strictly unique product, it does give an easy way to identify two products. This is what was going on when we made the choices of "red" and "blue" in our sum. A choice had to be made, but the choice itself wasn't especially important.

In fact, our definitions of $\times$ and $+$ are products and coproducts respectively, for finite sets. Further, almost all of our results (associativity, commutativity, units, ...) typically go through for other realms. However, distributivity doesn't usually hold.

2. Coherence Laws

2.1. The Worry. Since, as we have seen, we often need to make arbitrary choices when applying our definitions, can we be sure that our choices are somehow consistent?

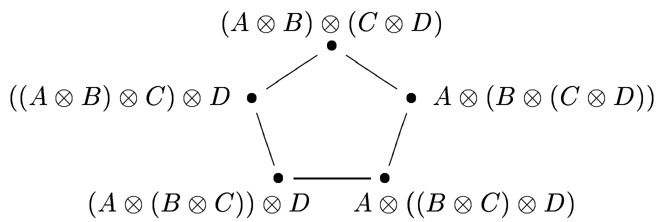
For example, when we study the tensor product, we wish to identify $A \otimes (B \otimes C)$ with $(A \otimes B) \otimes C$ and use either for a 3-fold tensor product. We usually make an identification

$$a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c.$$

Is it ok identify

$$a \otimes (b \otimes c) \mapsto -(a \otimes b) \otimes c$$

instead? Consider the following sequence of reassociations:



If we use the second identification, we'll find that

$$a \otimes (b \otimes (c \otimes d)) = (-1)^5 a \otimes (b \otimes (c \otimes d)),$$

which is a rule that we don't usually wish to impose on a tensor product.

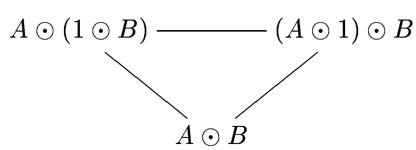
Before going on, I want to mention that the previous shape is called the *4th associahedron*. It parameterizes the possible reassociations of a 4-fold product. We think of the pentagon as being "filled in" if going around either way gives the same result. Notice that the 3rd associahedron is a line segment - in fact, we're using 5 copies to make up the edges of the pentagon. I've built a model of the 5th associahedron.

Definition. An *A_∞-space* X is a topological space with sequence of multiplications $\mu\colon X^n \to X$ whose reassociations are continuously parameterized by the associahedra.

Theorem. A connected space is homotopy equivalent to a loop-space if and only if it is a A_∞-space.

2.2. Mac Lane's Coherence Theorem.

Theorem (Mac Lane)**.** Suppose we are in a context with a multiplication, $\odot$, and unit, 1, which satisfy the above pentagon identity and makes



commute. Then any diagrams which we make from $\odot$, the unit, the associator, and identities will commute.

So associativity isn't that bad - we only have to verify that the pentagon identity is satisfied and then the rest of the associahedra are automatically ok.

Question. But what about coherence for our multiplication being commutative?

This one is harder - sometimes we don't want all diagrams to commute. First off, we shouldn't expect $A \odot B$ to be equal to $B \odot A$, but perhaps just isomorphic.

Definition. We call our multiplication *braided* if we have a twisting isomorphism:

$$A \odot B \longleftrightarrow B \odot A.$$

If twisting twice equals the identity, then we call the multiplication *symmetric*.

Example. The Cartesian product for finite sets is symmetric. But notice, $A \times B \neq B \times A$.

Example. The tensor product for vector spaces is symmetric using the usual twist map. We could instead twist

$$u \otimes v \mapsto 7v \otimes u,$$

in which case the multiplication is not symmetric. Indeed, twisting twice has the effect of multiplying by 49.

Example. Consider a different context: objects are lists of dots, functions are strings, and the product is concatenation. Then the product is braided, but not symmetric.

One can state and prove theorems along the lines of Mac Lane's dealing with symmetric or braided products. But it is often difficult to decide which coherence laws are "correct" and what we expect of a coherence theorem.

2.3. **A Non-algebraic Approach to Coherence.** So I want to take a different approach to coherence. I think I'll introduce it by using a conversation amongst friends that may sound familiar:

Question. What do you want for dinner tonight?

answer 1: I'd like a burrito.

answer 2: Food.

Notice the power in being non-committal - you avoid having to make an arbitrary choice. Of course, in real life this type of response gets old quite quickly.

We call a theory *algebraic* if it is of the first type - we make arbitrary decisions by specifying a favorite. A theory is *non-algebraic* if we refuse to commit to a choice.

We've already encountered a good example of this distinction - I originally defined the sum of sets using "red" and "blue" coloring, an obviously arbitrary decision. I went back later and enlarged my definition of the sum of sets to be any set that smelled like my original definition, and we gave a way to identify any two sums.

Although my second definition of the sum of sets was non-algebraic, it was still *biased* - that is I defined how to take the sum of two sets and then expected that you could use that definition inductively to fill in what an n-fold sum ought to be. There will be issues of associativity to work out. One could instead specify all n-fold products and how they should be compatible. This would be an *unbiased* approach.

I want to end by giving a non-algebraic non-biased definition of the tensor product of vector spaces. It is inspired by a class of objects called "opetopes."

We will fill a bucket with shapes:

(1) We'll throw a single point in - the field.
(2) We'll throw in a line segment for each vector space.
(3) For each n-linear map $V_1, \ldots, V_n \to W$, we'll throw in a shape like this:

FIX ME. *picture here.*

Notice that the dual picture is a tree. It's then easy to see how to compose the multilinear maps:

FIX ME. *picture here.*

Definition. *niche* and *occupant*.

Definition. We say that an occupant is *universal* if any other occupant factors uniquely through.

Definition. Our bucket is called an *opetopic set* if

(1) Every niche has a universal occupant.
(2) Compositions of universal occupants are universal.

In fact, our bucket is an opetopic set.

Definition. A *tensor product* is a universal occupant for a niche of vector spaces.

So this gives us an unbiased non-algebraic definition of tensor product. A similar sort of thing works for products and sums where we use other buckets of opetopes.

What does this have to do with coherence? Let's take a look at associativity for our tensor product:

FIX ME. *picture*

In this picture, we take a tensor product of V and W and then we take a tensor product of $V \otimes W$ with X. This has the same shape as a niche for a 3-fold tensor product of V, W, and X. As compositions of universals are universal, this tells us that our choice of $(V \otimes W) \otimes X$ is in fact a choice of product $V \otimes W \otimes X$. Not only that, the 2-dimensional shapes tells us how to use the tensor products.

In fact, in our non-algebraic, non-biased approach, we don't have to worry about coherence. It's somehow built into the shapes we use and our non-committal attitude.

References

[1] John Baez, *An Introduction to n-Categories*. 1997.

[2] John Baez, *Lectures on n-Categories and Cohomology*. 2007.

[3] John Baez and James Dolan, *Categorification*. 1998.

[4] John Baez and James Dolan, *From Finite Sets to Finite Diagrams*. 2000.

[5] John Baez and James Dolan, *Higher-Dimensional Algebra III: n-Categories and the Algebra of Opetopes*. 1997.

[6] John Baez and James Dolan, *Higher-Dimensional Algebra and Topological Quantum Field Theory*. 1995.

[7] Eugenia Cheng and Aaron Lauda, *Higher-Dimensional Categories: an illustrated guide book*. Draft, June 2004.

[8] Bernhard Keller, *Introduction to A-Infinity Algebras and Modules*. March, 2003.

[9] Tom Leinster, *Higher Operads, Higher Categories*. Cambridge University Press, Cambridge, 2004.

[10] Jacob Lurie, *What is an ∞-Category?*. Notices of the AMS, Sept 2008.

[11] Saunders Mac Lane, *Categories for the Working Mathematician*. Springer, 1978.

[12] Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in Algebra, Topology, and Physics*. American Mathematical Society, 2000.