

n -CATEGORIES PART I: A GLOBULAR APPROACH TO HIGHER CATEGORIES

JOHN FOLEY

1. STRICT n -CATEGORIES

Warning. We will write composition from left to right, top to bottom, back to front, etc. In particular, we write $((x)g)f$ or just xgf .

Recall that a strict 2-category is a category \mathcal{C} which has

- for all objects $A, B \in \text{Obj}_{\mathcal{C}}$, we require $\text{Hom}(A, B)$ be a category;
- bifunctors

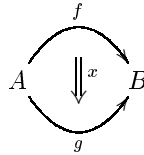
$$\circ: \text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$$

called compositions;

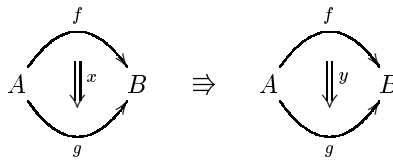
- unit, associativity, and interchange laws.

We could follow this definition to get a definition of n -category through enrichment. We're going to generalize, instead, globularly. That is, we'll develop a consistent language of glob pictures to believe in; we'll live with the algebra.

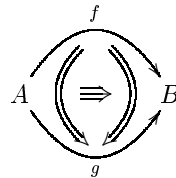
We draw a 2-morphism as:



and a 3-morphism as



or



. This should remind you of our favorite CW-structure for the k -disk and should tell you how to draw an arbitrary k -morphism with higher dimensional pencil and paper. We will use k -globs to picture k -morphisms in an n -category.

Let us briefly explain the combinatorial underpinnings of a globular n -category, analogous to the graph of an ordinary category.

notes by Ben Hummon.

Definition. An n -globular set is a graded set with maps:

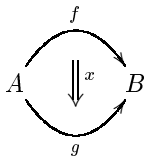
$$\begin{array}{c} X_n \\ s \downarrow \quad \downarrow t \\ X_{n-1} \\ s \downarrow \quad \downarrow t \\ \vdots \\ s \downarrow \quad \downarrow t \\ X_0 \end{array}$$

We enforce a couple of conditions:

$$xss = xts$$

$$xtt = xst.$$

Here, s and t should be read as *source* and *target*. The relations should look reasonable when considering the picture:



Here, xss and xts both refer to A .

Remark. Let \mathbb{G}_n be the category generated by

$$\begin{array}{c} n \\ \sigma_n \uparrow \quad \uparrow \tau_n \\ n-1 \\ \sigma_{n-1} \uparrow \quad \uparrow \tau_{n-1} \\ \vdots \\ \sigma_1 \uparrow \quad \uparrow \tau_1 \\ 0 \end{array}$$

and rules:

$$\sigma_{n-1} \sigma_n = \sigma_{n-1} \tau_n$$

$$\tau_{n-1} \tau_n = \sigma_{n-1} \tau_n$$

So a globular set is just a contravariant functor $\mathbb{G}_n \rightarrow \text{Set}$.

Definition. Let A_\bullet be an n -globular set. We define the fiber product,

$$\begin{aligned} A_m \times_{A_p} A_m &= \{(x, x') \in A_m \times A_m \mid xt^{m-p} = x's^{m-p}\} \\ &= m\text{-globs that we can glue along } p\text{-globs.} \end{aligned}$$

Definition. A *strict* n -category is a globular set with composition maps

$$\circ_p: A_m \times_{A_p} A_m \rightarrow A_m, \quad 0 \leq p < m \leq n.$$

and an injection

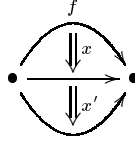
$$i: A_p \hookrightarrow A_{p+1}.$$

We write $(x)i$ as 1_x or, perhaps better, ${}_x1$. These satisfy axioms: $(x, x', x'', y, y' \in A_m)$

Source 1: if $p = m - 1$, then

$$(x \circ_p x')s = xs.$$

For the \circ_1 composition of 2-morphisms,

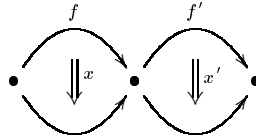


this refers to f .

Source 2: if $p < m - 1$, then

$$(x \circ_p x')s = (xs) \circ_p (x's).$$

For the \circ_0 composition of 2-morphisms,

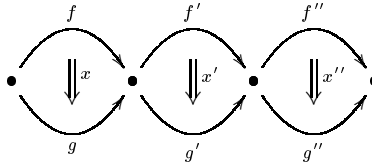


this refers to $f \circ_0 f'$.

Associativity:

$$(x \circ_p x') \circ_p x'' = x \circ_p (x' \circ_p x'').$$

For example, this axiom tell us that the 3-fold composition

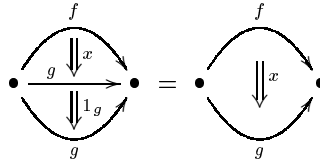


is well defined and saves us from horrible diagrams disgraced by gaudy decorations.

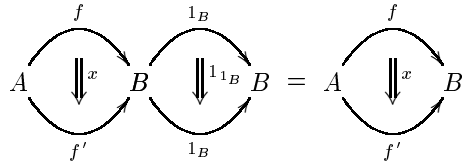
Identity 1:

$$x \circ_p (x' s^{m-p} i^{m-p}) = x.$$

For composition of 2-morphisms, this gives



and



Identity 2:

$$1_{x \circ_p x'} = 1_x \circ_p 1_{x'}.$$

For example, we have

Interchange:

$$(x \circ_p x') \circ_q (y \circ_p y') = (x \circ_q y) \circ_p (x' \circ_q y').$$

For example,

We have only listed the “source” axioms for s , but we require similar “target” axioms for t as well.

Fact. Doing strict n -categories via enrichment is the same as doing them globularly.

Example. The category of strict $(n-1)$ -categories is a strict n -category.

Example. There is a free strict n -category generated by one morphism C_i in each dimension. Informally, this looks like:

0-morphisms: Just C_0 .

1-morphisms: $1_{C_0}, C_1, C_1 \circ_0 C_1, C_1 \circ_0 C_1 \circ_0 C_1$, etc.

2-morphisms: $1_{1_{C_0}}, 1_{C_1}, C_2, C_2 \circ_0 C_2, C_2 \circ_1 C_2, (C_2 \circ_1 C_2) \circ_0 C_2$, etc.

etc:

In some sense, this example enumerates the ways that one can glue globular cells together in a strict n -category.

2. EXAMPLES OF n -CATEGORIES

Example (Top). The ∞ -category of topological spaces.

0-morphisms: spaces X, Y .

1-morphisms: maps $X \rightarrow Y$.

2-morphisms: homotopies of maps.

3-morphisms: homotopies of homotopies of maps.

etc:

This is strict up to level 2, then becomes weak. Weak inverse for $k > 1$.

Example. Fix a space Y .

0-morphisms: points in Y , which we’ll think of as maps $* \rightarrow Y$.

1-morphisms: paths in Y , which we’ll think of as maps $[0, 1] \rightarrow Y$.

2-morphisms: homotopies of paths into Y relative to the boundary.

etc:

This example has weak inverses on all levels.

Definition. We call the previous example the *fundamental ω -groupoid* of Y and write $\Pi_n(Y)$.

Example (Physics). We expect cobordisms of manifolds with corners to have a weak n -category structure. This is important for studying TQFTs.

Remark. There are also examples from logic and functional programming that I don’t understand as well.

3. WEAK n -CATEGORIES

The Yoneda-like strictifying lemma for weak 2-categories doesn't work for $n \geq 3$. See the following talk for more on this. So it is important to define a weak version of an n -category.

Comment from Ben: my notes aren't so strong through this last section of the talk. I believe John is following Tom Leinster's book and Eugenia Cheng's survey of definitions of n -category. Basically, we start with the free ω -category generated by one morphism in each level as an unlabeled, unbiased, strict ω -category. We then allow bias, add labels, and chop down to an n -category. This is all done in the language of generalized operads.