n-CATEGORIES PART II: THE BAEZ-DOLAN PERIODIC TABLE

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1. Many Levels

All n-categories in this talk will be assumed to be weak. Further, there is no agreed upon definition of n-category for $n \geq 4$, so most of this talk is wild conjecture. Following John's talk, I will draw globular pictures for compositions. We will follow the work of several authors (see the references), but especially John Baez.

Definition. An $\left(\begin{smallmatrix} n \\ \cdot \\ k \end{smallmatrix}\right)$-*category* is an $(n+k)$-category for which there is a single j-morphism, $0 \leq j < k$.

We can draw a schematic diagram for the levels of a $\left(\begin{smallmatrix} 3 \\ \cdot \\ 2 \end{smallmatrix}\right)$-category as:

external level		internal level
5	m	3
4	m	2
3	m	1
2	m	0
1	s	
0	s	

An s stands for "single" or "stupid;" an m stands "many."

Definition. An *internal j-morphism* is a $(j+k)$-morphism. An *internal object* is an internal 0-morphism.

The dot in our notation between n and k is meant to remind us of this internal objects level, which isn't counted by n nor by k. The level of a morphism is meant to indicate the dimension of the shape we draw. Our pictures for $\left(\begin{smallmatrix} n \\ \cdot \\ k \end{smallmatrix}\right)$-categories may decide to use either external or internal dimensions.

Let me remind you what compositions we have at each level of an n-category:

level	compositions
n	$\circ_0, \circ_1, \ldots, \circ_{n-1}$
$\vdots$	$\vdots$
2	$\circ_0, \circ_1$
1	$\circ_0$
0	

The subscript indicates the dimension of the subspace that we're gluing over.

Definition. The compositions $\circ_j$ will be called

$$\begin{cases} \textit{ambient} & \text{if } 0 \leq j < k \\ \textit{internal} & \text{if } k \leq j < n. \end{cases}$$

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Example. Recall Dan Rogalski's example of bicategory:

level	morphisms	compositions
2	linear maps	$\otimes$, $\circ$
1	bimodules	$\otimes$
0	rings	

If we fixed one ring, then we would have a $\left(\begin{smallmatrix}1\\ \cdot\\ 1\end{smallmatrix}\right)$-category and $\otimes$ would be ambient.

2. The Periodic Table

$\left(\begin{smallmatrix}n\\ \cdot\\ k\end{smallmatrix}\right)$-categories have other names for small n and k. The periodic table[3] lets us keep track of these:

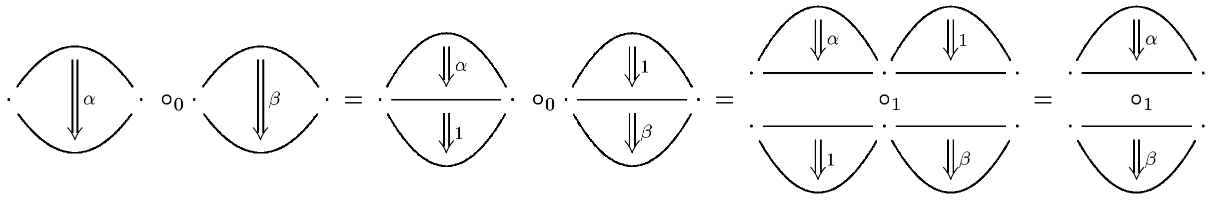
$k\backslash n$	0	1	2
0	sets	categories	2-categories
1	monoids	monoidal categories	monoidal 2-categories
2	commutative monoids	braided monoidal categories	braided monoidal 2-categories
3		symmetric monoidal categories	sylleptic monoidal 2-categories
4			strongly involutory mon 2-cats

2.1. The Zeroth Column.

$\left(\begin{smallmatrix}0\\ \cdot\\ 0\end{smallmatrix}\right)$**-category:** This is just a set of objects.

$\left(\begin{smallmatrix}0\\ \cdot\\ 1\end{smallmatrix}\right)$**-category:** A category with only one object is a monoid. The multiplication of elements is given by composition of morphisms.

$\left(\begin{smallmatrix}0\\ \cdot\\ 2\end{smallmatrix}\right)$**-category:** A 2-category with only one object and only one morphism is a commutative monoid. Where does commutativity come from? Consider the following Eckmann-Hilton diagram:



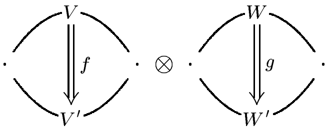
In this picture, α and β have made a quarter turn around each other. This shows that, in some sense, horizontal and vertical composition are the same. Further, by continuing on an other quarter twist, we see that the composition is commutative.

2.2. The First Column.

$\left(\begin{smallmatrix}1\\ \cdot\\ 0\end{smallmatrix}\right)$**-category:** This is just a plain, old category.

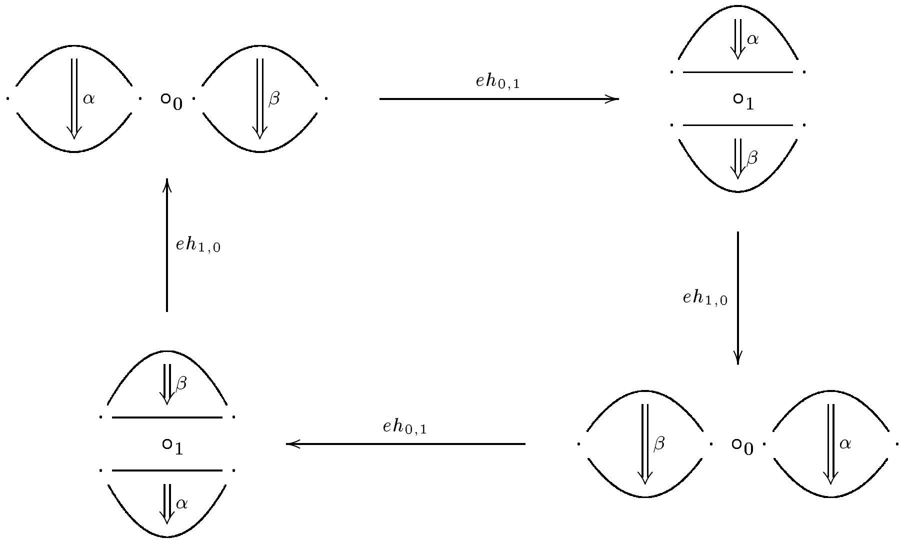
$\left(\begin{smallmatrix}1\\ \cdot\\ 1\end{smallmatrix}\right)$**-category:** This is a monoidal category. Tensoring of internal objects and their morphisms comes from the $\circ_0$ compositions. We get composition of internal morphisms from $\circ_1$.

$$\cdot \overset{V}{\rule{3em}{0.4pt}} \cdot \otimes \cdot \overset{W}{\rule{3em}{0.4pt}} \cdot$$



$\left(\begin{smallmatrix}1\\ \cdot\\ 2\end{smallmatrix}\right)$-category: A $\left(\begin{smallmatrix}3\\ \cdot\\ 2\end{smallmatrix}\right)$-category is a braided monoidal category.

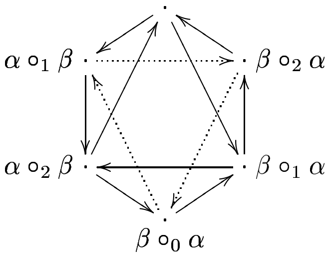
The Eckmann-Hilton diagram for a $\left(\begin{smallmatrix}1\\ \cdot\\ 2\end{smallmatrix}\right)$-category still holds, except that we no longer demand equalities, but rather internal isomorphisms. We get two Eckmann-Hilton maps which we can use to circle two globs around each other:

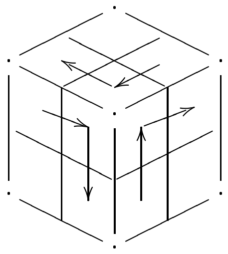


Penon's (reflexive) globular definition of n-category has too many coherence laws in that it includes a rule that makes this square commute[5].

$\left(\begin{smallmatrix}1\\ \cdot\\ 3\end{smallmatrix}\right)$-category: This is a symmetric monoidal category. In a symmetric monoidal category, we require that the previous square commute. Where does this relation come from?

Proceeding as before, we get an octahedron, where the edges are Eckmann-Hilton maps. The globs are now football-shaped, and drawing them glued across 1 or 2 dimensional faces is tricky. Notice that there are 3 squares in the octahedron, which are copies of the twisting from before. We want to show that these squares commute, and we'll do it by showing that the triangular faces commute.





Looking at a triangle in a dual picture, we can watch the cell α travel around a 2x2x2 Rubik's cube. One might convince oneself that the Eckmann-Hilton maps have no effect when traveling around the triangle.

2.3. **About the Table.** Notice that as we move down a column, we expect more structure. In the zeroth column, we start with a set. Then we equip the set with a multiplication. And then we require that multiplication be commutative. A similar process, but more drawn out, happens in the first column. We start with a category. Then we introduce a multiplication $\otimes$ on objects. In the next row, we require that $x \otimes y$ be isomorphic to $y \otimes x$. In the last row, we require that this isomorphism squares to the identity. A similar pattern happens in the second column.

Conjecture (Baez and Dolan)**.** The columns of the periodic table stabilize for $k \geq n + 2$.

This would say, in some sense, that commutative monoids are the nicest form of $\left(\begin{smallmatrix}0\\ \cdot\\ k\end{smallmatrix}\right)$-category and symmetric monoidal categories are the nicest form of $\left(\begin{smallmatrix}1\\ \cdot\\ k\end{smallmatrix}\right)$-category.

3. Braiding

Definition. Let n-Braid$_k$ denote the free $\left(\begin{smallmatrix}n\\ \cdot\\ k\end{smallmatrix}\right)$-category generated by a single internal object[2].

Example. 1-Braid$_2$ is the braided monoidal category with levels:

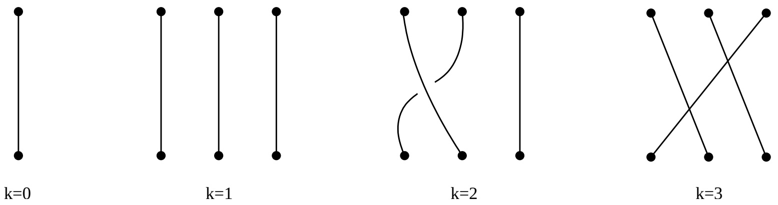
level	morphisms
3	necessary coherence
2	$\langle x \rangle$, free on $\circ_0$, $\circ_1$
1	s
0	s

Here is a table of n-Braid$_k$ for low n and k:

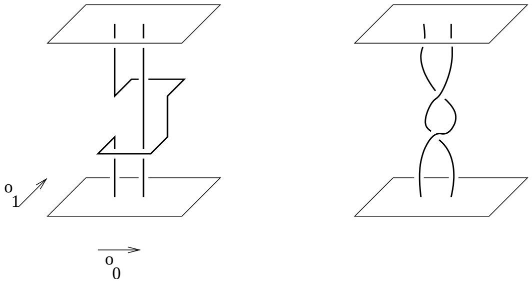
$k \backslash n$	0	1
0	singleton set	stupid category
1	$(\mathbb{N}, +, 0)$ as a monoid	objects are tensor powers; only 1 morphisms
2	$(\mathbb{N}, +, 0)$ as comm monoid	braid groupoid
3		symmetric groupoid

The stupid category, 1-Braid$_0$, has a single object and only its identity morphism. We can use Mac Lane coherence to strictify 1-Braid$_1$. This ensures that we only get identity morphisms. For higher k, though, we get more morphisms.

Typical morphisms in 1-Braid$_k$:



We might also draw a morphism in 1-Braid$_2$ as follows:



A horizontal slice in this picture contains the ambient compositions. The vertical direction is the internal composition. Writing two dots - internal objects - next to each other in a slice is meant to indicate their $\circ_0$ composition. These pictures are good when the ambient compositions are free - they are less helpful otherwise. To draw these pictures, we are relying heavily on our compositions commuting - that is, we have an interchange law.

We think of n-morphisms in n-Braid$_k$ as braids with n-dimensional strands sitting in $n + k$ dimensions. If we instead work with n-categories with duals, then we get tangles, in the style of Ben's talk.

4. Homotopy Theory

The periodic table might remind us of the following table of homotopy groups of spheres: $\pi_{n+k}(S^k)$

$k\backslash n$	0	1	2
0	$\pi_0(S^0) \cong \{N, S\}$	$\pi_1(S^0) \cong 1$	$\pi_2(S^0) \cong 0$
1	$\pi_1(S^1) \cong \langle x \rangle$	$\pi_2(S^1) \cong 0$	$\pi_3(S^1) \cong 0$
2	$\pi_2(S^2) \cong \mathbb{Z}$	$\pi_3(S^2) \cong \mathbb{Z}$	$\pi_4(S^2) \cong \mathbb{Z}/2$
3		$\pi_4(S^3) \cong \mathbb{Z}/2$	$\pi_5(S^3) \cong \mathbb{Z}/2$
4			$\pi_6(S^4) \cong \mathbb{Z}/2$

Indeed, we know π_0 is a set, π_1 is a group, and π_2 is abelian. This feels quite similar to the zeroth column of the periodic table. Also, each of these columns have stabilized by the Fruedenthal suspension theorem.

Recall from John's talk:

Definition. The *fundamental ω-groupoid* of a space X is the one given by

0-morphisms: points in X.
1-morphisms: paths between points.
2-morphisms: homotopies between parallel 1-morphisms.
higher: etc.

We denote it by $\Pi_\omega(X)$. By taking isomorphism classes at level n, we get the *fundamental n-groupoid*, $\Pi_n(X)$.

Π_ω is a functor

$$\text{Top} \longrightarrow \omega\text{-Grpd}$$

We also have a functor,

$$|\cdot| \colon \omega\text{-Grpd} \longrightarrow \text{Top},$$

called *geometric realization.* It patches together cells to make a space out of a groupoid.

These functors are adjoint, but they don't give an equivalence of categories. However, following Grothendiek[1], we expect these functors to extend to ω-functors, setting up an equivalence of ω-categories. Under this premise, homotopy n-types correspond to n-groupoids.

Let (X, x_0) be a space with basepoint. How are $\pi_j(X, x_0)$ and $\Pi_j(X)$ related? A typical element of $\Pi_2(X)$ looks like:

$$\begin{tikzcd}
x \arrow[r, bend left=50, "f"{name=F}] \arrow[r, bend right=50, "g"'{name=G}] & y \arrow[Rightarrow, from=F, to=G]
\end{tikzcd}$$

up to a homotopy fixing the boundary. If we insist that $x = y = x_0$ and $f = g = 1_{x_0}$, then we collapse the boundary. This gives a sphere, up to homotopy, relative to the basepoint. We can generally recognize $\pi_j(X)$ as a subgroup (or sub-object) of $\Pi_j(X)$, restricting to top-level morphisms on $1_{\cdots 1_{x_0}}$.

What does moving down one row of the periodic table look like in homotopy-theory-world? Well, we want to get a new space in which we've pushed each of the homotopy groups up one level. So it sounds like we want suspension.

Now we can intuitively see what the table of homotopy groups of spheres has to do with the period table. If we made a table of fundamental n-groupoids, $\Pi_{n+k}(\Sigma^k S^0)$, then they should live in the periodic table. Instead, we're taking π_{n+k}, which captures some of that information.

5. Moving Around the Periodic Table

Definition. Given parallel j-morphisms, f, g, $\mathrm{Hom}(f, g)$ is an $(n-j)$-category with object set $\{f, g\}$, called a *microcosm*.

We have a bunch of ways to move around the periodic table: (the arrow in parentheses shows which direction the operation moves on the periodic table.)

decategorification (←):

$$\begin{tikzcd}
4 & m & \\
3 & m \arrow[r] & m \\
2 & m \arrow[r] & m \\
1 & s & s \\
0 & s & s
\end{tikzcd}$$

Identify morphisms in the $(n-1)$ level which are isomorphic according to the top level. (Take isomorphism classes.)

discrete categorification (→):

$$\begin{tikzcd}
4 & & m \\
3 & m \arrow[r] & m \\
2 & m \arrow[r] & m \\
1 & s & s \\
0 & s & s
\end{tikzcd}$$

Add a top level of identities. By repeatedly discretely categorifying, we can regard any n-category as an ω-category.

forgetting monoidal structure (↑):

$$\begin{tikzcd}
4 & m \arrow[dr] & \\
3 & m \arrow[dr] & m \\
2 & m \arrow[dr] & m \\
1 & s & m \\
0 & s & s
\end{tikzcd}$$

Ignore the lowest (monoidal) level, by say, taking a microcosm category.

stabilization (↓):

$$\begin{tikzcd}
5 & & m \\
4 & m \arrow[ur] & m \\
3 & m \arrow[ur] & m \\
2 & m \arrow[ur] & s \\
1 & s & s \\
0 & s & s
\end{tikzcd}$$

This is a conjectured left adjoint to "forgetting monoidal structure." The stabilization conjecture states, more precisely, that this suspension functor is some sort of equivalence for $k \geq n+2$.

delooping (↗):

$$\begin{tikzcd}
4 & m \arrow[r] & m \\
3 & m \arrow[r] & m \\
2 & m \arrow[r] & m \\
1 & s \arrow[r] & m \\
0 & s & s
\end{tikzcd}$$

Think of the highest monoidal level as a "many" level.

looping (↙):

$$\begin{tikzcd}
4 & m \arrow[r] & m \\
3 & m \arrow[r] & m \\
2 & m & s \\
1 & s & s \\
0 & s & s
\end{tikzcd}$$

Let a be *the* k-morphism. Take the microcosm subcategory $\mathrm{Hom}(1_a, 1_a)$, keeping the monoidal levels. That is, we're looking at the loops on 1_a.

6. Strictification

Theorem. Every 2-category is biequivalent to a strict one.

Proof. (sketch[7]) Let $[\mathcal{C}, \mathcal{D}]$ denote the collection of weak functors from $\mathcal{C}$ to $\mathcal{D}$. We embed a weak 2-category $\mathcal{C}$ via a weak functor

$$Y : \mathcal{C} \to [\mathcal{C}^{op}, \mathrm{Cat}].$$

It sends an object A of $\mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(-, A)$. ☐

The previous theorem can be seen as a generalization of the Mac Lane coherence theorem that Joel presented. Indeed, a monoidal category is a 2-category.

Question. Why don't we just strictify n-categories and be done with it?

Answer. We can't. A weak $\left(\begin{smallmatrix}1\\ \cdot\\ 2\end{smallmatrix}\right)$-category has braiding, but a strict $\left(\begin{smallmatrix}1\\ \cdot\\ 2\end{smallmatrix}\right)$-category is symmetric. Equivalence of braided categories preserves symmetry. So because 1-Braid$_2$ has braiding, it isn't equivalent to a strict 3-category. Nor is $\Pi_3(S^2)$. This, presumably, has something to do with the Hopf fibration, which generates $\pi_3(S^2)$.

Definition. A *Gray n-category* is a n-category in which everything is strict, except interchange[4].

Theorem. Every 3-category is equivalent to a Gray 3-category.

Remark. We don't entirely like Gray categories, because Steven Lack has proved, that the tricategory of bicategories is not triequivalent to the tricategory of Gray 2-categories.

Joyal and Kock[6] have a theorem that relate homotopy 3-types to 3-groupoids which are strict in all ways, except for identity arrows.

Answer. We have ways two ways to make 3-categories semi-strict. People seem to think that the Joyal-Kock method should work for higher n, too.

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